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Organizers:

Ker-I Ko (New York),
Anil Nerode (Cornell),
Klaus Weihrauch (Hagen).

Foreword

All over the world numerous computers are used for real number computation. They evaluate real functions, find zeroes of functions, determine eigenvalues and integrals and solve differential equations, and so they perform or at least are expected to perform computations on sets like the set of real numbers, the set of open subsets of real numbers or the set of differentiable real functions. The increasing demand for reliable as well as fast software in scientific computation and engineering requires a sound and broad foundation. *Computable analysis* is the mathematical theory of those functions on the real numbers and other sets from analysis, which can be computed by machines. It connects the two classical disciplines analysis/numerical analysis and computability/complexity theory combining in particular the central concepts of limit and approximation on the one hand and of machine models and computation on the other hand. Computable analysis may serve as an additional framework for numerical analysis and all other disciplines which need an exact concept of computation for real functions.

Though computable analysis started in the early years of computability theory, the field is still in its infancy. It has a great potential for further development, since there are numerous challenging open problems, many basic questions have not yet been studied systematically and only occasionally its concepts have been applied to advanced problems.

The second Dagstuhl seminar on Computability and Complexity in Analysis was attended by 36 scientists from 11 countries. The 25 talks presented at the seminar mirror the present state of the art.

We express our gratitude to the staff of Schloss Dagstuhl for perfect organization and providing a great atmosphere.

Abstracts

Variants of Computable Analysis and Realizability

Andrej Bauer
Carnegie Mellon University, Pittsburgh, USA

There are several schools of computable analysis, among others: recursive analysis, type II computability, effective domains, effective T_0-spaces, effectively given continuous domains, effective equilogical spaces, and Blum-Smale-Shub Real Numbers Machine.

In this talk I show that all of these variants of computable analysis arise as, or are closely related to, realizability constructions over various partial combinatory algebras. This gives the subject of computable analysis a more uniform outlook. It also makes available the tools from category theory and categorical logic, which are traditionally not employed very much in the study of computable analysis.

We can interpret the results of John Longley about logical full abstraction for programming languages as saying that to each programming language corresponds a school of computable analysis. Thus, one's choice of a particular variant of computable analysis should depend on what programming language one intends to use to implement algorithms. In many ways I find this view satisfactory because it is pragmatic and it explains where all the variants of computable analysis come from.

An Application of Domain Representations

Jens Blanck
University of Wales Swansea, U.K.

Streams and transformations on streams are very common in Computer Science, especially as models of time dependant functions. We model streams as functions from time to data. Both time and data may be either discrete or continuous. We build representations of stream spaces from representations of time and data. Since streams naturally are discontinuous we need to consider approximative representations to get computability. Representations of stream transformers have also been considered.

Recursive Operations over Topological Structures

Vasco Brattka
FernUniversität Hagen, Germany

We present a definition of recursive multi-valued operations over topological structures. One of the main results states that over a certain class of structures, so-called perfect structures, recursive operations coincide with computable operations. Moreover, perfect structures uniquely characterize their computability theory. Finally, we define classes of recursive sets over structures and we show that these notions are generalizations of the classical notions from recursion theory and computable analysis.

Π^0_1 Classes and Index Sets in Computable Analysis

Douglas Cenzer
University of Florida, USA
(joint work with Jeff Remmel)

Index sets are defined for effectively closed sets and for computably continuous functions. The complexity of various problems in the arithmetical hierarchy is determined. For example, we consider the problem of finding the measure and cardinality of a set and of the difference of two sets. We consider the problem of testing whether two functions are equal (which is Pi-0-2 complete) or differ by less than some fixed amount. We also consider the problem of whether one function is the derivative of another and the problem of whether a given function has a computably continuous derivative (which is Sigma-0-3 complete). An effective version of the Closed Graph Theorem for the Baire Space is shown.

A New Foundation for Computational Geometry and Solid Modelling

Abbas Edalat
Imperial College, London, U.K.
André Lieutier
Matra Datavision & XAOLab, LIM, Marseilles, France

Correctness of algorithms in computational geometry is usually proved using the unrealistic Real RAM machine model of computation with the undesirable result that correct algorithms, when implemented, turn into unreliable programs.

We use a domain–theoretic approach to recursive analysis to develop the basis of an effective and realistic framework for solid modeling. This framework is equipped with a well–defined and realistic notion of computability which reflects the observable properties of real solids. It is closed under Boolean operations, admits non–regular sets and supports a design methodology for actual robust algorithms.

Within this model, some unavoidable limitations of solid modeling computations are proved and a sound framework to design specifications for feasible modeling operators are provided which can incorporate existing methods into a general, mathematically well–founded theory. Moreover, the model is able to capture the uncertainties of input data in actual CAD situations.

On Effective Metric Spaces and Representations of the Reals

Armin Hemmerling
Ernst-Moritz-Arndt–Universität Greifswald, Germany

An effective metric space (EMS) is a triple (X, S, d) with a complete metric space (X, d) and a "skeleton" $S = (s_n)_{n \in \mathcal{N}}$ being a sequence of elements which is dense in (X, d) and for which the set $\{\langle m, n, k\rangle : d(s_m, s_n) < q_k\}$ is recursively enumerable, where $(q_k : k \in \mathcal{N})$ is a standard numbering of the rational numbers. Very often the set $\{\langle m, n, k\rangle : d(s_m, s_n) > q_k\}$ is recursively enumerable too; then we are speaking on a strongly effective metric space (SEMS). These notions go essentially back to Lacombe, Moschovakis, Weihrauch et al..

Modifying the Ko–Friedman approach, approximate computability of (partial) functions $f : X \rightarrowtail X'$ from an EMS to another one is defined by means of oracle Turing machines which process natural numbers corresponding to the indices of points of the skeletons. Some usual consequences are sketched, and a criterion of equivalence of different skeletons in a metric space is easily obtained.

The topological arithmetical hierarchy gives a useful tool to classify point sets in EMSs. So the domains of computable functions are just the Π_2^{ta} sets of this hierarchy, i.e., the recursive G_δ sets.

Representations of the real numbers are surjective functions $\varrho : X \rightarrowtail \mathcal{R}$, where X is equal to $\mathcal{F} = \mathcal{N}^{\mathcal{N}}$ or to $\mathcal{B} = \{0, 1\}^{\mathcal{N}}$. A representation is said to be standard if it is computationally equivalent to the normed Cauchy representation. It turns out that the standard representations are just those functions ϱ, for which there are computable representations $\underline{\varrho}$ and $\overline{\varrho}$ satisfying $\underline{\varrho} \subseteq \varrho \subseteq \overline{\varrho}$ and having inversions which are computable (as relations) in a natural sense. In particular, the computable standard representations are just those computable functions which have computable inversions.

Nonlinear Integration Problems of High Average Complexity

Peter Hertling
Fernuniversität Hagen, Germany

We analyse the complexity of nonlinear Lebesgue integration problems in the average case setting for continuous functions with the Wiener measure and the complexity of approximating the Itô stochastic integral. Wasilkowski and Woźniakowski (1999) studied these problems, observed that their complexities are closely related, and showed that for certain classes of smooth functions with boundedness conditions on derivatives the complexity is proportional to ε^{-1}. Here $\varepsilon > 0$ is the desired precision with which the integral is to be approximated. They showed also that for certain natural function classes with weaker smoothness conditions the complexity is at most of order ε^{-2} and conjectured that this bound is sharp. We show that this conjecture is true.

Real Functions Computable by Finite Automata using Incremental Representations

Michal Konečný
University of Birmingham, U.K.

We try to classify the functions of the type $I^n \to I$ (for some interval $I \subseteq \mathcal{R}$) that can be realized by finite automata. Such a class strongly depends on the choice of real number representation. We consider only so-called IFS representations where numbers are represented by sequences of digits and each of the digits is a contraction on I.

First we study the special case of IFS representations in which all digits are affine contractions. It turns out that every finitely computable function in this setting must be affine on every region where it is continuously differentiable. Conversely, it is easy to see that every function whose graph is composed of (parts of) hyperplanes with rational coefficients can be finitely computed using the signed binary representation which is an affine representation.

In the case of more general IFS representations we can get a similar limitation result for finitely computable functions using bijections T that translate a contraction d to an *affine* contraction $T \circ d \circ T^{-1}$.

The author finds it difficult to say much about these translators T in general, but in the case of LFT (=Möbius transformations) representations (studied by Edalat, Potts, Heckmann and others) we can arrive at a complete LFT version of the affine result: Any function of the type $I^n \to I$ which is finitely computable using an LFT representation is equal to an LFT on every region where it is continuously differentiable.

Solving Differential Equations

David Lester
Manchester University, U.K.

In my talk I discuss some of the practical problems associated with solving differential equations in a computable analysis setting. These include the following:

- An exact series solution of a PDE might not converge fast enough to be useful; and
- Initial Value Problems require a Lipschitz condition to be computable; and
- Function representation has an important rôle to play in the efficiency of an algorithm; and
- Even though the effective Weierstraß Theorem can compute an approximate polynomial for a computable function, it takes too long to be practical.

If you know otherwise, email me: dlester@cs.man.ac.uk !

Finite Approximations of Metric Spaces

Henri Lombardi
Université de Franche-Comté, Besançon, France

We give some examples for supporting the need of uniform versions of theorems of analysis from the bit-complexity point of view.

First we examine the case of real roots of real polynomial. The pointwise theorem says that the roots of a polynomial with polynomial time computable real coefficients are polynomial time computable. But the fact is that the set of real roots is not recursively computable. So the pointwise theorem is strange, and false in the intuitive meaning. We get a uniform theorem saying that some set of “real or virtual” real roots is uniformly computable in polynomial time from the coefficients. Here it is necessary to extend the set of real roots in order to get a continuous variation of this set. Once this is done (see [GLM98]), the uniform theorem is much better than the pointwise one, since it says something about the general computation of roots for any real polynomial.

Second we deal with the polynomial time Weierstrass approximation theorem of Hoover, and we explain how to obtain, using the Hoover's techniques, a uniform theorem that gives a uniform result for all real functions (on a compact interval) instead of giving a result only for polynomial time computable real functions. (see [LLM97])

Third, we discuss the computation of complex roots of a complex monic polynomial. The result is usually stated as: there is an algorithm that computes the list of the roots. In fact this is not a uniform theorem since there is no way of computing continuously such a list. The uniform version deals with a function from monic polynomials to multisets of complex numbers.

This discussion leads to the following issue. In many natural situations, when we compute a compact metric space (e.g. the set of roots of a monic complex polynomial) we do not obtain uniformly any point of the compact space, but we obtain finite approximations of the space "from outside". So the usual description of compact metric spaces via skeletons (that means approximating the compact space from inside) seems not well adapted to uniform computations. Approximating metric spaces by finite sets with rational distances between points should be investigated systematically in order to get better uniform theorems. This is in the same spirit of the Richman's claim about the need of a development of constructive analysis without countable choice, in order to get better and nicer theorems (abstract of [RicTfa] see also [RicGen]: Can constructive mathematics be developed in a reasonable manner without the axiom of countable choice? Serious schools of constructive mathematics all assume it one way or another, but the arguments for it are not compelling. Here it is shown how the fundamental theorem of algebra can be restated and proved without using countable choice, and it is argued that this is really the right way to look at it. A notion of a complete metric space, suitable for a choiceless environment, is also developed.).

[GLM98] González-Vega L., Lombardi H., Mahé L. *Virtual roots of real polynomials*, J. Pure And Applied Algebra **124**, (1998) 147–166.

[LLM97] Labhalla S., Lombardi H., Moutai E. *Espaces métriques rationnellement présentés et complexité, le cas de l'espace des fonctions ré-*

elles uniformément continues sur un intervalle compact, to appear in Theoretical Computer Science.

[RicTfa] Richman F. *The fundamental theorem of algebra: a constructive development without choice*, (21 pages, 16 October 1998)

[RicGen] Richman F. *Generalized real numbers in constructive mathematics*, Indagationes Mathematicae, **9** (1998), 595-606.

Using logic to design efficient algorithms

Klaus Meer
TU Chemnitz, Germany
(joint work with J.A. Makowsky)

We introduce a new sparsity condition on multivariate polynomials in n variables (over some ring R) and show that under this condition many otherwise intractable problems involving these polynomials become solvable in polynomial (even linear) time in n (in the BSS-model over R). To define our sparsity condition we associate with these polynomials a hypergraph and study classes of polynomials where this hypergraph has tree width at most k for some fixed $k \in \mathcal{N}$. Our method uses graph theoretic and model theoretic tools developed in the last 15 years and applies them to the algebraic setting.

A Note On How To Compute Multi-valued Functions

Norbert Müller
Universität Trier, Germany

Multi-valued functions are of increasing interest at least in the field of imperative programming languages for exact real arithmetic. We compare two notions of computability for these multi-valued functions: 'cylinder-computability' allowing precise recursive characterizations and 'single-path-computability' which is better suited for implementations.

Lower Bounds on Discrepancy and the Complexity of Integration

Erich Novak
Universität Erlangen, Germany

Can we compute $I_d(f) = \int_{[0,1]^d} f(x)\, dx$ for $f : [0, 1]^d \to \mathbf{R}$ from F_d in polynomial time, i.e.,

$$\mathrm{cost}(\varepsilon, F_d) \le C \cdot \varepsilon^{-\gamma} \cdot d^{\beta} \; ?$$

The answer clearly depends on the classes F_d, there are three types of results:

- intractability results: for certain F_d no polynomial time algorithms exist, the (normalized) L_2-discrepancy is intractable
- tractability results by probabilistic reasoning: the L_∞-discrepancy is tractable
- tractability results by constructive methods: the weighted case and Wiener integrals.

We use the real number model and know that it is enough to consider quadrature formulas $Q_n(f) = \sum_{i=1}^n a_i\, f(x_i)$. For the L_2-discrepancy we know from Roth (1954, 1980) and Frolov (1980) that

$$\inf_{t_i} \mathrm{disc}_2(\{t_1, \ldots, t_n\}) \asymp n^{-1} \cdot (\log n)^{(d-1)/2},$$

hence the order does not depend on d. Nevertheless the problem is intractable,

$$n(\varepsilon, F_d) \ge C^d \,(1 - \varepsilon^2)$$

with some $C > 1$. We also study weighted norms and weighted discrepancy, the weight of the ith variable is w_i. Results for $F_{d,w}$ depend on the weights as follows:

The problem is tractable iff $\limsup_d \sum_{j=1}^d w_j / \log d < \infty$;

$n(\varepsilon, F_{d,w}) \le C \cdot \varepsilon^{-2}$ iff $\sum w_j < \infty$.

Results of this type for positive quadrature formulas are due to Sloan, Woźniakowski (1998).

The main result concerning the L_∞-discrepancy (or star-discrepancy) is

$$c \cdot d \, \log \varepsilon^{-1} \le n(\varepsilon, F_d) \le C \cdot d \, \varepsilon^{-2},$$

i.e., the complexity is linear in the dimension d. Up to now it was not known whether

$$n(1/3, F_d) \leq C \cdot 3^d.$$

The talk is based on two papers from N., Woźniakowski (1999) and from Heinrich, N., Wasilkowski, Woźniakowski (1999).

Effective Subsets of Metric Spaces

Gero Presser
Universität Dortmund, Germany

In my work I generalised some results that have been published by Brattka and Weihrauch in 1999 for the Euclidean space (cf. V. Brattka and K. Weihrauch: *Computability on subsets of Euclidean spaces I: closed and compact subsets*, Theoretical Computer Science 219, 1999).

I introduce some representations of the closed subsets of a computable metric space. Some of these turn out to be equivalent (with respect to computable reduction). For others to be equal, the space must have some additional properties.

I deal with the same kind of question on the space of compact subsets of computable metric spaces.

Extended Admissibility

Matthias Schröder
FernUniversität Hagen, Germany

We give a new definition of admissible representations which allows to handle topological spaces which are not second–countable.

We show that admissible representations δ_X, δ_Y of topological spaces X, Y have the property that a partial function $f :\subseteq X \to Y$ is continuously realizable w.r.t. δ_X, δ_Y (i.e. there is a continuous function $\Gamma :\subseteq \Sigma^\omega \to \Sigma^\omega$ with $\delta_Y \circ \Gamma = f \circ \delta_X$), iff f is sequentially continuous.

Furthermore, the class of the spaces having an admissible representation is proven to equal the class of the T_0–spaces with a countable pseudobase. Here, a pseudobase of a topological space X is a set $\mathcal{B} \subseteq 2^X$ such that for all open sets O, all $x \in O$ and all sequences (y_n) converging to x there is a set $B \in \mathcal{B}$ such that $x \in B \subseteq O$ holds and the sequence (y_n) is eventually in B.

Many interesting operators creating new topological spaces from old ones are shown to preserve the existence of an admissible representation (e.g. cartesian product and exponentiation).

Thus, a reasonable computability theory is possible even on important non–second–countable spaces.

An Intrinsic Topology for (all) Sets

Dana S. Scott
Carnegie Mellon University, Pittsburgh, PA, USA

Any set can be given either the discreet or the indiscreet topology, but these are trivial and not interesting. In set theory, it is usually agreed that every entity is a set of some sort, and, hence, every set can always be regarded as a set of sets. As a consequence of this convention, every set is a subset of a powerset. Now powersets have two intrinsic product topologies: one making them compact Hausdorff spaces and a weaker topology that only satisfies T_0 separation. We concentrate on the second. It has many properties (well known from different presentations). For example, the intrinsic T_0 topology can produce a homeomorphic example of every topological space. Moreover, there is a simple way to describe continuous mappings between sets in set-theoretical terms. Thus, sets and continuous mappings give an equivalent category to the topological category of T_0 spaces. A new category, EQU, consists of (arbitrary set-theoretical) equivalence relations and continuous equivariant mappings. This category has some surprising properties including being cartesian closed.

Representations Versus Numberings: Some New Results

Dieter Spreen
Universität Siegen, Germany

This paper gives an answer to Weihrauch's question (Weihrauch, K., *Computability*, Springer, Berlin, 1987) whether and, if not always, when an effective map between the computable elements of two represented sets can be extended to a (partial) computable map between the represented sets.

Examples are known showing that this is not possible in general. A condition is introduced and for countably based topological T_0-spaces it is shown that exactly the (partial) effective maps meeting the requirement are extendable. For total effective maps the extra condition is satisfied in the standard cases of effectively given separable metric spaces and continuous directed-complete partial orders, in which the extendability is already known. In the first case a similar result holds also for partial effective maps, but not in the second.

The Kolmogorov Complexity of Real Numbers

Ludwig Staiger
Martin-Luther-Universität Halle-Wittenberg, Germany

The talk focussed on the following three topics.

1. Classes of real numbers definable by properties of base r expansions
2. Application of Kolmogorov complexity to the calculation of Hausdorff dimension
3. Right computability of Hausdorff dimension yields a small recursive presentation of a G_δ-cover of an Σ_2-definable set

For a real number $\alpha \in [0, 1]$, we consider the Kolmogorov complexity of its expansions with respect to different bases and we show that the length of the $l \cdot \log_r b$ prefix of the base r expansion of α is the same (up to an additive constant) as the $log_r b$-fold complexity of the length l prefix of its base b expansion.

Using this base independence of Kolmogorov complexity we derive relationships between various classes of real numbers, such as random, Borel normal, disjunctive, Liouville and computable numbers to the classes of complexity defined numbers.

Moreover, we present the above mentioned connections to the estimation of the Hausdorff dimension of sets of real numbers.

Abstract versus Concrete Computation on Topological Partial Algebras

J.V. Tucker
University of Wales Swansea, U.K.
(joint work with J.I. Zucker, McMaster University)

Let A be a many sorted algebra. Abstract models of computation on A define finite computations that are independent of any representations of A and are isomorphism invariant. Concrete models of computation on A define computations via some representation $r : R \to A$. We review these models in the case that A is a topological algebra. Roughly speaking the abstract models define a single class of computable functions $Abstract(A)$ and, similarly, the concrete models also define a single class of computable functions (up to equivalent representations) $Concrete(A, r)$. We consider the relationship between them. Usually, $Abstract(A)$ is a subset of $Concete(A, r)$ and the question arises for what A and r does

$$Abstract(A) = Concrete(A, r)?$$

We survey results for while-array computation on real numbers (TCS Vol. 219 (1999)) and announce new results for metric algebras. We extend the while language with a countable choice (cc) instruction to define many valued functions. We explain the need for these functions through basic examples (pivot, approximation). We show that for f an effectively locally uniformly continuous function on A, and certain general conditions on r, f is while_{cc}–computable if, and only if, f is r-computable.

Computability and Delta function

Masako Washihara
Kyoto Sangyo University, Japan

We discuss the computability of Dirac's delta function by the function space approach. This approach was first introduced by Pour-El and Richards. In [1], they proposed the concept of computability structure on Banach spaces. This concept was extended to Fréchet spaces in [2].

The space $\mathcal{S}$ of all rapidly decreasing functions is a Fréchet space by the norm system $\|f\|_m = \|(N+1)^m f\|$ where $N = \frac{1}{2}(x^2 - D^2 - 1)$ and $\|\cdot\|$ denotes the L^2-norm. The intrinsic computability structure on $\mathcal{S}$ is given as follows. A sequence $\{f_n\}$ in $\mathcal{S}$ is *computable* if :

(i) $\{x^k D^j f_n\}$ is computable as a triple sequence of continuous functions on R, and

(ii) $\exists d$; recursive, $|x| \geq d(k,j,n,N)$ implies $|x^k D^j f_n(x)| \leq 2^{-N}$.

An effective generating set in $\mathcal{S}$ is the sequence of Hermite finctions:

$$\phi_n(x) = (-1)^n \pi^{-\frac{1}{4}} \sqrt{2^n n!} e^{\frac{1}{2}x^2} D^n(e^{-x^2}).$$

We have $(N+1)\phi_n = (n+1)\phi_n$, and therefore, $\|\phi_n\|_m = (n+1)^m$.

The delta function δ is a tempered distribution, that is, a continuous linesr functional on $\mathcal{S}$. In [3], we discussed the computability in the sace $\mathcal{S}'$ of all tempered distributions. This space can be considered as the union of an increasing sequence of Banach spaces by the following method.

Let $\mathcal{S}_m$ be the completion of $\mathcal{S}$ by the norm $\|\cdot\|_m$ and $\mathcal{S}'_m$ be its dual. Then the following statements holds:

(I) $\mathcal{S}' = \cup_{m=1}^{\infty} \mathcal{S}'_m$.

(II) Each $\mathcal{S}'_m$ is a Banach space by the norm $\|u\|_{-m} = sup\{|u(f)| : f \in \mathcal{S}, \|f\|_m \leq 1\}$.

(III) $\mathcal{S}'_m \subset \mathcal{S}'_{m+1}$ and $\|u\|_{-m} \geq \|u\|_{-(m+1)} \quad (u \in \mathcal{S}'_m)$.

(IV) $u_n \to u$ in $\mathcal{S}'$ iff $\exists m; u_n \to u$ in $\mathcal{S}'_m$.

Every $f \in \mathcal{S}$ is in $\mathcal{S}'$ by putting $f(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx \quad (\phi \in \mathcal{S})$. Then we have $\|f\|_{-m} = \|(N+1)^{-m} f\|$. In particular, $\|\phi_n\|_{-m} = \frac{1}{(n+1)^m}$.

We define effective convergence in $\mathcal{S}'$ and $\mathcal{S}'$-computability as follows: $\{u_{nk}\}$ is said to converge to $\{u_n\}$ in $\mathcal{S}'$ effectively if $\exists d$; recursive, $u_{nk}, u_n \in \mathcal{S}'_{d(n)}$, and $\|u_{nk} - u_n\|_{-d(n)}$ converges to 0 effectively. $\{u_n\}$ is called $\mathcal{S}'$-computable if there exists an $\mathcal{S}$-computable double sequence $\{f_{nk}\}$ which converges to $\{u_n\}$ in $\mathcal{S}'$ effectively.

Then our results are:

Proposition 1 *If $\{u_n\}$ is $\mathcal{S}'$-computable, then $\exists d; recursive$, $\{\|u_n\|_{-d(n)}\}$ is computable.*

Proposition 2 *If $\{u_n\}$ is $\mathcal{S}'$-computable and $\{f_n\}$ is $\mathcal{S}$-computable, then $\{u_n(f_m)\}$ is computable.*

Proposition 3 *If $\{u_n\}$ is $\mathcal{S}'$-computable, then $\{D^j u_n\}$ is $\mathcal{S}'$-computable.*

Proposition 4 *Dirac's delta function δ is $\mathcal{S}'$-computable.*
The sequence $\{D^j \delta\}$ is $\mathcal{S}'$-computable

References.

[1] Marian B.Pour-El and J. Ian Richards, Computability in Analysis and Physics, Springer-Verlag (1989)

[2] M. Washihara, Computability and Fréchet spaces, Mathematica Japonica, vol.42 (1995)

[3] M. Washihara, Computability and tempered distributions, Mathematica Japonica, vol.50 (1999)

Upper Bounds on Discrepancy and the Complexity of Integration

Henryk Woźniakowski
Columbia University, USA, and University of Warsaw, Poland

In this talk we consider the classical L_2- and L_∞-discrepancy of n points in the d dimensional unit cube. It is well known that the discrepancy measures the worst case error of linear algorithms for multivariate integration over the unit ball of the tensor product Sobolev space with regularity one. The complexity of integration is therefore, roughly, the inverse function of the discrepancy.

We present explicit upper bounds on the discrepancy for the absolute and normalized cases. The main emphasis is on the dependence on d. In particular, we mention a recent result of S. Heinrich, E. Novak, G. Wasilkowski and the author that the L_∞-discrepancy is bounded by $C\sqrt{d/n}$ for some absolute constant C.

Computability problems of discontinuous functions
Functional space approach

MARIKO YASUGI
Kyoto Sangyo University, Japan

I am to discuss how to view notions of computability for *discontinuous functions*. I confine myself to real-valued functions from some spaces.

In what follows, I have worked with V. Brattka, T. Mori, Y. Tsujii and M. Washihara.

Our standpoint in studying computability problems in mathematics is *doing mathematics*. That is, we would like to talk about computable functions and other mathematical objects just as one talks about continuous functions, integrable functions, etc.

In any naive notion of computability of a function (on a compact set), *uniform continuity* is inherent. On the other hand, one often *approximates* discontinuous functions, for instance in numerical computations and drawing graphs. Very often such approximations are successful and satisfactory. It is therefore meaningful and important to speculate on computability of discontinuous functions.

According to our standpoint, it is a *mathematical* investigation to formulate computability of discontinuous functions and to find out how it is related to some existing mathematical notions.

One possible *method* of such investigations is to work in some theories of *abstract spaces*; for example, functional spaces such as Banach spaces and Frechet spaces, metric spaces, uniform topological spaces. The reason why such a method is effective for the purpose is that an abstract theory of spaces supplies us with a logical framework for mathematics. A functional space is defined axiomatically, and the objects which satisfy such axioms are those that are *well-controlled*.

For example, $L^1[0,1]$-space contains continuous functions, but many more. An object in this space is controlled by the property that it be not too notched so that the "area" can be determined. Computability in such a space means a good (effective) approximation by "naive" computable objects so that the area can be nicely approximated. So, if we place a little bit more restriction on the objects than just integrability, we can characterize computable objects. Pour-El& Richards thus introduced an axiom system which characterizes (sequences of) computable objects in a Banach space. One notices that it is the axiom system of a Banach space restricted to "computable" objects. On can extend the same idea to other functional spaces.

The domain of a function needs not be a subset of real numbers. We thus extend our consideration to more general metric spaces. Here too, one can adopt the axiomatic approach.

Metrization of a Banach space preserves computability of the metric, while it has not been settled if a fourth metrization of a Frechét space preserves computability.

Although a uniform topological space with a countable index set is equivalent to a metiric space, we notice that, in defining computability structure, metric itself is not necessary. It is the uniformity of the base system that is used in describing the computability structure.

From such a standpoint, we have investigated comutability structures on abstract spaces as well as computability problems of specific objects.

Interpolation functor and computability

Atsushi Yoshikawa
Graduate School of Mathematics, Kyushu University, Japan

For a *compatible* couple of Banach spaces $\mathbf{X}$, $\mathbf{Y}$, Calderón's complex interpolation space $[\mathbf{X}, \mathbf{Y}]_\theta$ is defined as the image of the evaluation mapping $f \mapsto f(\theta)$ from an auxiliary space $\mathcal{F}(\mathbf{X}, \mathbf{Y})$ to the space $\mathbf{X} + \mathbf{Y}$. $\mathcal{F}(\mathbf{X}, \mathbf{Y})$ consists of the *essentially* bounded holomorphic functions $f = f(z)$ on the strip $0 < \Re z < 1$, taking values in $\mathbf{X}$ and in $\mathbf{Y}$ on the lines $\Re z = 0$ and $\Re z = 1$, respcetively,

Combine this situation with Pour-El's notion of computability in Banach spaces. Assuming each of $\mathbf{X}$ and $\mathbf{Y}$ admits a computability structure, raise these computability structures to the spaces $\mathcal{C}(\mathbf{X})$ or $\mathcal{C}(\mathbf{Y})$ of $\mathbf{X}$- or $\mathbf{Y}$-valued continuous functions on the real line, with good boundary behaviors at infinities.

It is a routine matter to verify that these computable structures induces a computable structure $\mathcal{S}$ in the auxiliary space $\mathcal{F}(\mathbf{X}, \mathbf{Y})$. Denote by $\mathcal{S}_\theta$ the image of the evaluation mapping $f \mapsto f(\theta)$ of the computability structure $\mathcal{S}$.

Theorem *Suppose* $\mathbf{X}$ *and* $\mathbf{Y}$ *admit computability structures in the sense of Pour-El and Richards. Let* θ *be computable,* $0 < \theta < 1$. *Then* $[\mathbf{X}, \mathbf{Y}]_\theta$ *admits a computability structure, in fact,* $\mathcal{S}_\theta$.

Weakly Computable Real Number and Closure Properties

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A real number x is weakly computable if there is a computable sequence $(x_n)_{n\in\mathcal{N}}$ of rational numbers which converges to x weakly effectively, namely $\lim_{n\to\infty} x_n = x$ and $\sum_{n=0}^{\infty} |x_n - x_{n+1}|$ is finite. It is shown by Weihrauch and Zheng that the class of weakly computable real numbers is closed under the arithmetical operations. Here we show that this class is not closed under the "effective limits" and the computable real functions. The key step to prove these result is that we can construct a non-ω-r.e. set A such that $x_A := \sum_{i\in A} 2^{-i}$ is weakly computable and that there is a computable real function which maps such x_A to x_{2A} while the later is not weakly computable any more by a result of Klaus Ambos . These results are also extended to a larger class of ω-weakly computable real numbers.

Computability Theory of Generalized Functions

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We propose a computability theory for generalized functions. While for countable sets there is a single computability theory (ordinary recursion theory); for real functions and other functions from analysis there are several mutually non-equivalent computability concepts (e.g., Bishop and Bridges, Blum, Shub and Smale, Grzegorczyk, Ko, Pour-El and Richards, Traub, Wasilkowski and Wozniakowski, Kreitz and Weihrauch, et cetera). One of these concepts is based on Turing's definition of computable real numbers and Grzegorczyk's definition of computable real functions. Along this line there are several different but consistent approaches (e.g., Edalat, Ko, Pour-El and Richards, Weihrauch, et cetera). We extend two of these models, the effective approximation approach (Pour-El and Richards) and the representation approach (Weihrauch), to test functions and generalized functions, Schwartz functions and tempered distributions, and distributions with compact support. While the first one, given in familiar analytic terms, provides for an interaction between the recursion theory and the theory of generalized functions, the second offers algorithms for performing computations of generalized functions on Type 2 Turing machines. The two approaches give rise to the equivalent computability theory for generalized functions. The computability of several basic operations on generalized functions are discussed as well as some applications, including an application to computable analysis of the Korteweg-de Vries Equation posted on the whole line.