Dagstuhl-Seminar 01461 on

Computability and Complexity in Analysis

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Organizers:

Vasco Brattka (Hagen), Peter Hertling (Hagen),
Mariko Yasugi (Kyoto), Ning Zhong (Cincinnati).

Preface

This meeting was the third Dagstuhl seminar which was concerned with the theory of computability and complexity over the real numbers. This theory, which is built on the Turing machine model, was initiated by Turing, Grzegorczyk, Lacombe, Banach and Mazur, and has seen a rapid growth in recent years. Recent monographs are by Pour-El/Richards, Ko, and Weihrauch.

Computability theory and complexity theory are two central areas of research in theoretical computer science. Until recently, most work in these areas concentrated on problems over discrete structures. In the last years, though, there has been an enormous growth of computability theory and complexity theory over the real numbers and other continuous structures. One of the reasons for this phenomenon is that more and more practical computation problems over the real numbers are being dealt with by computer scientists, for example, in computational geometry, in the modelling of dynamical and hybrid systems, but also in classical problems from numerical mathematics. The scientists working on these questions come from different fields, such as theoretical computer science, domain theory, logic, constructive mathematics, computer arithmetic, numerical mathematics, analysis etc.

This Dagstuhl seminar provided a unique opportunity for 46 participants from such diverse areas and 12 different countries to meet and exchange ideas and knowledge. One of the topics of interest was foundational work concerning the various models and approaches for defining or describing computability and complexity over the real numbers. We gained new insights into the computability-theoretic side of various computational questions from physics as well as from other fields involving computations over the real numbers. The 39 talks also covered fields closely connected to computable analysis such as recursion theory, algorithmic information theory, constructive mathematics, realizability theory, domain theory, proof theory, complexity theory and interval analysis. Last but not least, new implementations of exact real arithmetic and further developments of already existing software packages have been discussed.

We would like to thank all participants for their contributions and we express our gratitude to the staff of Schloß Dagstuhl for excellent organization!

Talks

On the Existence Theorems of Kantorovich, Moore and Miranda

Götz Alefeld
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It is shown that the assumptions of the Kantorovich theorem imply the assumptions of the Moore test. Using furthermore a result of Shen Zuhe and the author it can be proved that the Moore test always implies the successful application of the Miranda theorem. The conclusion is that the Kantorovich theorem implies the Miranda theorem. The converse is not true which can be shown by a counterexample.

A Relationship between Domain Theory and Type Two Effectivity

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In this paper I compare two well studied approaches to topological semantics–the domain-theoretic approach, exemplified by the category of countably based equilogical spaces, Equ, and Type Two Effectivity, exemplified by the category of Baire space representations, $\mathsf{RepN}^{\mathsf{N}}$. These two categories are both locally cartesian closed extensions of countably based T_0-spaces. A natural question to ask is how they are related.

First, we show that $\mathsf{RepN}^{\mathsf{N}}$ is equivalent to a full coreflective subcategory of Equ, consisting of the so-called 0-equilogical spaces. This establishes a pair of adjoint functors between $\mathsf{RepN}^{\mathsf{N}}$ and Equ. The inclusion $\mathsf{RepN}^{\mathsf{N}} \to \mathsf{Equ}$ and its coreflection have many desirable properties, but they do not preserve exponentials in general. This means that the cartesian closed structures of $\mathsf{RepN}^{\mathsf{N}}$ and Equ are essentially different. However, in a second comparison we show that $\mathsf{RepN}^{\mathsf{N}}$ and Equ do share a common cartesian closed subcategory that contains all countably based T_0-spaces. Therefore, the domain-theoretic approach and TTE yield equivalent topological semantics of computation for all higher-order types over countably based T_0-spaces.

Weierstrass Approximation Theorems for NP Real and #P Real Functions

Markus Bläser
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A result by Hoover states that each Ko–Friedman polynomial time computable function $f : [0, 1] \to \mathbb{R}$ can be approximated by a (weakly) polynomial times computable sequence (φ_n) of univariate polynomials up to an error of 2^{-n}, that is, for all n, $|f(x) -$

$\varphi_n(x)| \leq 2^{-n}$ for all $x \in [0,1]$. This can be viewed as a polynomial time version of the classical Weierstrass Approximation Theorem. In this talk, we discuss how similar variants of the Weierstrass Approximation Theorem can be obtained for the classes $\mathrm{NP}_{C[0,1]}$ and $\#\mathrm{P}_{C[0,1]}$.

Computable Versions of the Closed Graph and the Open Mapping Theorem

Vasco Brattka
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The open mapping theorem, Banach's inverse mapping theorem and the closed graph theorem are basic theorems of functional analysis whose classical proofs rely on the Baire category theorem. Therefore these theorems count as non-constructive. We discuss several computable and partially computable versions of these theorems which cannot be expressed in the same way in constructive analysis. The effective version of Banach's inverse mapping theorem states that on Banach spaces the inverse T^{-1} of any computable linear and bijective operator T is computable as well, whereas the inversion mapping $T \mapsto T^{-1}$ itself is computable only in the finite dimensional case. As an application of the effective version of the theorem we discuss the initial value problem of linear differential equations.

The Constructive Theory of Apartness Spaces

Douglas S. Bridges
University of Canterbury—Christchurch, New Zealand
(joint work with L.S. Vîţă, P.M. Schuster, H. Ishihara)

Apartness—between points and sets, and between sets and sets—is a generalisation of inequality. With the right axioms, the constructive theory of spaces endowed with a set–set apartness relation flows smoothly and provides a natural constructive substitute for classical topology. In this talk I first present the axioms for an apartness space and discuss the natural models: metric and uniform spaces. The main part of the talk will deal with continuity of functions, and with Cauchy nets and completeness, in the context of apartness spaces. Finding the right definition of a Cauchy net, and showing that for a sequence in a metric space this is constructively equivalent to the usual notion of a Cauchy sequence, turns out to be a non–trivial problem whose solution nicely illustrates some of the amusing aspects of proofs in constructive mathematics.

Effectively Closed Sets and Graphs of Computably Continuous Functions

DOUGLAS CENZER
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(joint work with JEFF REMMEL)

We compare and contrast four basic types of effectively closed sets C, where (i) the set of closed intervals which with nonempty intersection with C is recursively enumerable (r.e.), (ii) the set of closed intervals with empty intersection with C is r.e., (iii) the set of open intervals which with nonempty intersection with C is r.e., and (iv) the set of open intervals with empty intersection with C is r.e. It is shown that (i) implies (iii) and that (iv) implies (ii), but no other implications hold. In fact, the family of sets satisfying (ii) but not (iv) is Σ_4^0 complete. We compare the computability and complexity of a continuous real function F with the computability and complexity of the graph G of the function F. In particular, the graph always satisfies (ii) and (iii), but may satisfy any combination of the other two conditions. A similar analysis is carried out for functions on these fundamental subspaces of the real line: the Cantor space, the Baire space and the unit interval. Notions of polynomial time decidable and NP for closed sets are given, and a version of the P=NP question is posed for closed subsets of the Cantor space, which is shown to be equivalent to the standard P=NP question.

Reals and Randomness

ROD DOWNEY
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There are various notions of algorithmic randomness, such as Schnorr randomness, Martin-Löf randomness, Chaitin randomness etc. These are usually based on effective topology ("random means avoid all effectively null properties") or compressibility ("random strings are hard to compute"). We will look at the background in this area, and discuss recent results by the author together with Hirschfeldt, Laforte, Stephan, and Nies. In particular, we will look at recent work calibrating randomness, and a surprising solution to Post's problem.

"The" Interval

Martín Escardó
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(joint work with Alex Simpson)

Up to isomorphism, there is exactly one complete ordered field. It can be concretely implemented in set theory in various ways, e.g. via Dedekind cuts or via equivalence classes of Cauchy sequences of rational numbers. However, in practice, one works axiomatically, starting from an unspecified complete ordered field as the real-number system.

Thus, in principle, we know without a shadow of a doubt what the real line is. The trouble is that there are many categories around other than that of sets—for instance, that of topological spaces. There, one usually imposes the Euclidean topology on the set of real numbers. But where does this topology come from? Is this choice arbitrary? Or is it forced upon us?

Many categories were considered in this seminar, including: (1) Sets and continuous functions. (2) Topological spaces and continuous maps. (3) Baire-represented spaces and realizably continuous maps. (4) Baire-represented spaces and realizably computable maps. (5) Natural-number-represented sets and recursively realizable functions. (6) Equilogical spaces. (7) Toposes with natural-number objects. (8) Domains.

For each of these categories, there are natural choices of real-numbers object. (1) and (2) have already been discussed. (3) The set-theoretical reals with any admissible representation. (4) Same with any admissible representation making the four basic operations computable and the inequality relation $x < y$ semi-decidable. (5) The recursive real numbers (a countable field this time). (6) Signed-digit binary expansions under the Cantor topology and the obvious equivalence relation. (7) Here there are two different competing candidates: the Dedekind and Cauchy reals, which coincide only in particular examples of toposes, such as ones satisfying countable choice over the natural numbers. In general, the latter is a subobject of the former. (8) Two choices, which are used for different purposes, have been considered here: the set-theoretical reals under their natural order, and the domain of closed and bounded intervals of reals under reverse inclusion.

Thus, it appears that we have a proliferation of notions of real line: at least one for each category. One of the objectives of this work is to clarify the situation, showing that, nevertheless, we are entitled to speak of "the" real line. A more important one is to provide a general notion of real line that applies to many mathematical and foundational settings. Among the mathematical settings, all the above categories and more should qualify. Among the foundational settings, we wish to include at least set theory, intuitionistic set theory, topos theory and type theory. An even more important objective is that the notion be geometrically motivated and have computational content.

For both technical and geometric reasons, we have chosen to look at closed-and-bounded line segments rather than the whole real line. We introduce an abstract notion of interval object, sketched below, which applies to any category with finite products. Such a category may or may not have an interval object, but, when it does, it is uniquely determined up to isomorphism. The intention is that an interval object is "the" interval in the category.

In order to achieve this, we work with abstract midpoint algebras, which are objects endowed with a binary "midpoint" operation subject to three equations that correspond to basic geometric properties of the bisection operation as performed with ruler and compass by the ancient Greeks. A source of examples in the categories of sets and topological spaces is n-dimensional Euclidean space, where any convex subset with the concrete midpoint operation is a midpoint algebra. Also, products of midpoint algebras with the pointwise operations are themselves midpoint algebras. We refer to a midpoint algebra subject to two further axioms as a(n abstract) convex object. The axioms are cancellation and what we call iteration. Roughly, iteration amounts to a completeness property—we refer the reader to our publications for its formulation and geometrical interpretation. We define an interval object to be a free convex object over two generators—if we denote the generators by 0 and 1, then we could denote the interval object by $[0, 1]$. Geometrically, the free property provides an affine path between any two points of a convex object. Computationally, it provides an analogue of primitive recursion for natural numbers. For instance, multiplication can be defined by the free property. Moreover, the basic algebraic laws of multiplication are easily established from the free property.

Returning to the examples of categories considered above, in (1)–(6) we get the proposed natural notions of interval. For (7) and (8), the situation is more subtle. (7) For toposes, neither of the competing objects is obtained: we get the Cauchy completion of the (potentially incomplete!) object of Cauchy reals within the (always complete) object of Dedekind reals - that is, the intersection of all Cauchy complete subobjects of the object of Dedekind reals that contain the Cauchy reals. Under countable natural-number choice, the three notions agree. (8) There is no interval object in the category of domains. However, the two typical real-number domains do arise as free convex objects: The interval under its natural order arises as the free convex object over the Sierpinski domain, and the domain of intervals arises as the free convex object over the domain with one minimal and two maximal elements.

For topological spaces, it is also natural to consider generating objects other than the two-point discrete space. For example, the free convex object over the Sierpinski space is the set-theoretical interval under the topology of lower semicontinuity, which is often considered in analysis. Generalizing the situation of the Euclidean interval, the free convex object over the $(n + 1)$-point discrete space is the n-simplex with the Euclidean topology.

In intuitionistic type theory, the usual notion of interval is obtained by considering the category of setoids.

We finish with three conjectures.

(i) By an application of the adjoint-functor theorem, we know that, in various categories of domains and continuous maps, the free convex object over any generating object exists. We conjecture that, in the category of continuous dcpos, it is given by the probabilistic power domain of the generating object.

(ii) Again via the adjoint-functor theorem, we know that in the category of compact Hausdorff spaces and continuous maps, the free convex object over any generating space exists. We conjecture that it is the space of Borel regular probability measures under the weak topology, with insertion of generators given by point measures. The free property would then amount to integration of convex-object-valued continuous maps.

(iii) We conjecture that the interval object in the category of locales and continuous maps is the usual localic interval. Moreover, we formulate this conjecture for locales over any base topos. What is interesting here is that the set of global points of the localic line is known to coincide with the object of Dedekind reals.

Computable Scalar Fields: a Basis for PDE Software

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(joint work with Helmer André Friis and Hans Munthe-Kaas)

Partial differential equations (PDEs) are fundamental in the formulation of mathematical models of the physical world. Computer simulation of PDEs is an efficient and important research tool in science and engineering. In this context we present the notions of scalar and tensor fields, and discuss why these abstractions are useful for the practical formulation of solvers for PDEs.

A scalar field is a function from some manifold, e.g., a unit cube in $\mathbf{R}^n$, to the real or complex numbers. It therefore has the pointwise lifting of operations on the reals (or complex numbers), specifically the field operations are defined for scalar fields. In addition, scalar fields have partial derivatives for the spatial directions, as well as volume and surface integration for subdomains of the manifold. Tensor fields are built on top of scalar fields and are the basis for formulating PDEs and PDE solvers. More about these abstractions and their use in PDE software can be found in *Scientific Programming* 8(4), a special issue devoted to coordinate-free numerics.

Given computable scalar fields, the operations on tensor fields will also be computable. As a consequence we get computable solvers for PDEs. The traditional numerical methods for achieving computability by various approximation techniques (e.g., finite difference, element or volume methods), all have artifacts in the form of numerical inaccuracies and various forms of noise in the solutions. What we need is a theory for computable scalar fields, which either lets us understand why this has to be so, or provides us with better tools for constructing these basic build blocks.

Approximate Decidability in Euclidean Spaces

Armin Hemmerling
Ernst-Moritz-Arndt–Universität Greifswald, Germany

We study concepts of decidability (recursivity) for subsets of Euclidean spaces $\mathbb{R}^k$ within the framework of approximate computability (type two theory of effectivity). A new notion of approximate decidability is proposed and discussed in some detail. It is an effective variant of F. Hausdorff's concept of resolvable sets and modifies and generalizes notions of recursivity known from computable analysis, formerly used for open or closed sets only, to more general types of sets. Approximate decidability of a set can equivalently be expressed by computability of the characteristic function by means of an appropriately working oracle Turing machine. The notion fulfills some further natural requirements and is invariant under canonical embeddings of sets into spaces of higher dimensions. However, it is not closed under binary union or intersection of sets. We also show how the framework of resolvability and approximate decidability can be applied to investigate concepts of reducibility for subsets of Euclidean spaces.

Random Sets versus Random Sequences

Peter Hertling
FernUniversität Hagen, Germany

Besides the classical randomness notion for infinite binary sequences one can define another randomness notion for sets of natural numbers. The relation between these two notions reminds one of the relation between recursive sets and recursively enumerable sets. It is easy to see that randomness of the characteristic sequence of a set implies randomness of the set and of its complement. We show that the converse is not true: there exists a set such that both the set and its complement are random but the corresponding characteristic binary sequence is nonrandom. This is a corollary of the following result about random sequences where we say that a sequence is contained in another sequence if the corresponding set is contained in the set corresponding to the other sequence: every random sequence is contained in a nonrandom sequence which is contained in a random sequence.

Relative 1-Randomness of Reals

Denis Hirschfeldt
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(joint work with Rod Downey and André Nies)

Following up on Rod Downey's talk, we give an example of the methods used in the study of the relative 1-randomness of computably enumerable reals. Specifically, we sketch the proof that every incomplete Solovay degree of c.e. reals is the join of two strictly smaller Solovay degrees of c.e. reals.

On the Power of Quantum Oracles

Elham Kashefi
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(joint work with Adrian Kent, Vlatko Vedral, and Konrad Banaszek)

A standard quantum oracle S_f for a function $f : Z_N \to Z_N$ is defined to act on two input states and return two outputs, with inputs $|i\rangle$ and $|j\rangle$ ($i, j \in Z_N$) returning outputs $|i\rangle$ and $|j \oplus f(i)\rangle$. For general f, this is the simplest invertible quantum map that allows the evaluation of any $f(i)$ with one call. However, if f is known to be a one-one, a simpler oracle, M_f, which returns $|f(i)\rangle$ given $|i\rangle$, shares these properties. We consider the relative strengths of these oracles. We define a simple promise problem which minimal quantum oracles can solve exponentially faster than classical oracles, via an algorithm which does not extend to standard quantum oracles. We then prove our main result: two invocations of M_f suffice to construct S_f, while $\Theta(\sqrt{N})$ invocations of S_f are required to construct M_f.

Proof Mining in Numerical Analysis:
Effective Strong Unicity in L_1-Approximation

Ulrich Kohlenbach
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(joint work with Paulo Oliva)

Proof mining denotes the project of extracting effective information out of prima-facie ineffective proofs by proof-theoretic techniques. We first give a short introduction to proof mining in general and show how this approach can be applied to a large class of ineffective uniqueness proofs for solutions of equations in the area non-linear (functional) analysis ([2]). The effective data extractable from such proofs are a-priori rates of so-called strong unicity which allow to compute the unique solution with arbitrary prescribed precision. In the second part of the talk we extract the first effective (in all parameters) rate of strong unicity for the best polynomial (of degree $\leq n$) L_1-approximation of functions $f \in C[0,1]$ from Cheney's proof of the uniqueness of that polynomial. Cheney's proof ([1]) is ineffective both by using classical logic as well as non-computable real numbers (in the form of the binary König's lemma WKL). The extracted rate of strong unicity has the optimal error dependency as follows from a result of Kroó ([4]).

References

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[3] Kohlenbach, U., Oliva, P., Effective bounds on strong unicity in L_1-approximation. Preprint, submitted (2001).

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Continuity of Non-Extensional Computation

Michal Konečný
University of Edinburgh, U.K.

We search for a version of the fundamental theorem of TTE for exhaustive non-extensional computation of a relation. In a situation with two spaces represented by admissible representations, we try to characterize the class of relations between the spaces that arise from continuous functions on the representing sequences. We show that for open representations this equals the class of lower-semi-continuous relations up to taking a closure on the image sets.

Semantic Characterisations of Computability over the Reals

Margarita Korovina
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(joint work with Oleg Kudinov)

In this talk we present an investigation in computability theory on the real numbers. Taking the domain theory approach as a starting point we develop computational models for higher type objects such as operators and real-valued functionals. Domain theory was introduced by Dana Scott as a mathematical theory of computation and is used to build semantics for programming languages.

We use effective ω-continuous domains for modeling infinite computational processes on the reals. This approach accords well with the intuitive notion of computability and closely related to the computable analysis. Our main subject of investigation is semantic characterisation of computable objects such as real-valued functions, functionals and operators. For this purpose we use an appropriate fixed-point logic with ability of expressing computable procedures. Generally, computation on the reals is an infinite process which produces approximations closer and closer to the result. In the framework of this approach an infinite computational process is characterised via the least fixed point of a Δ_0-operator.

One of the main features of this characterisation is that it does not depend on the choice of representation of the reals.

Such semantic characterisation could be viewed as a foundation for applications of model-theoretic methods and ideas to computable analysis.

Solving Differential Equations using Domains

Marko Krznarić
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(joint work with Abbas Edalat (Imperial College, London)
and André Lieutier (Dassault Systemes, France))

We use the recently developed domain theoretic version of Picard's theorem to provide a framework for solving differential equations with domains. The solution of a differential equation is approximated by a sequence of pairs of polynomials, one increasing and the other one decreasing, both tending to the function in the limit.

P-time Computable Reals and some Complexity Classes

Oleg V. Kudinov
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To analyse P-time computable reals we consider the class K of numerical total functions which are computable in polynomial time under the unary encoding of arguments and values. This class is closely related to our subject and it can be described in recursion-theoretic terms.

Starting from this, it is convenient to consider only reals in $[0, 1]$. The following theorem can be proved by the standard method of bisections.

Theorem. For any $\alpha \in [0, 1]$ α is P-time computable iff for some function $f \in K$ with values in the set $\{0\,,\,1\}$ the following presentation holds

$$\alpha = \frac{1}{2} + \sum_{k=2}^{\infty} \frac{2f(k) - 1}{2^k}.$$

Let E^2 denotes the second class of numerical functions from Grzegorczyk's hierarchy. It was A. Cobham who stated the inclusion $E^2 \subseteq K$ and considered the converse inclusion as a natural problem. Affirmative answer implies $\mathbf{P} \neq \mathbf{PSACE}$ and something more, but it could be negative. So, let class E consist of numerical functions which use less than $c\,|x| l(x)$ cells of the tape and $l(f(x)) \leq c\, l(x)$, where $|x|$ denotes the sum of integers from the tuple x. It is evident that $E^2 \subseteq E$ and the inclusion $K \subseteq E$ is of our interest. We need some definitions and special recursions.

A function f is defined from functions $g, \psi, k, \alpha_1, \ldots, \alpha_p$ by bounded return recursion (BRR) iff

- $f(\mathbf{x}, 0) = g(\mathbf{x})$
- $f(\mathbf{x}, y+1) = \psi(\mathbf{x}, y, f(\mathbf{x}, \alpha_1(\mathbf{x}, y)), \ldots, f(\mathbf{x}, \alpha_p(\mathbf{x}, y)))$
- $f(\mathbf{x}, y) \leq k(\mathbf{x}, y)$,

where $\alpha_i(\mathbf{x}, y) \leq y$ for $i = 1, \ldots, p$.
Lemma 1. The class K is closed under BRR and substitution.
Lemma 2. K is the smallest class closed under BRR and substitution and containing basic functions $0, s, +, \times, I_m^n$.

We say that a function f is defined from functions g, k by bounded sum ($f =$ BSUM(g, k))

$$\text{iff } f(\mathbf{x}, y) = \sum_{i=0}^{y} g(\mathbf{x}, i) \text{ } and \text{ } f(\mathbf{x}, y) \leq k(\mathbf{x}, y).$$

Let BRR(E^2) be the set of constructed from some elements of E^2 functions by BRR.
Remark. The class K contains BRR(E^2) and is closed under BSUM and substitution.
Lemma 3. K is the smallest class closed under BSUM and substitution and containing BRR(E^2).
Lemma 4. The class E contains BRR(E^2) and is closed under BSUM.

Probably, E is not closed under substitution and we can not state the inclusion $K \subseteq E$ yet. In fact, it is new interesting question and affirmative answer implies $\mathbf{P} \neq \mathbf{PSPACE}$.

References

[1] Ker-I Ko, Complexity theory of real functions, Progress in theoretical computer science, Boston, 1991.

Infinite Precision BLAS—(or why N. Müller is right)

David Lester
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(joint work with Adrian Tate)

In this brief talk we provide practical evidence that Norbert Müller is right: expanding the size of the basic operations in an exact arithmetic package with forward error propogation still doesn't provide sufficient efficiency to outperform a package with backward error propogation.

Type-2 Theory of Effectivity via Kleene's Function Realizability

PETER LIETZ
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Theorems of constructive mathematics can be transferred to results in TTE, such as computability or non-computability of certain functions and operations. We use Kleene's function realizability as a means of translation. For this purpose, it has to be verified that the realizability interpretations of descriptions of common spaces of analysis are in fact admissible representations of those spaces. Quite pleasantly, for a large class of examples, including test-functions and distributions, the interpretation of a description of the plain underlying set of a space turns out to be an admissible representation of that space equipped with the right topology. Distinct meaningful topologies on the same underlying set are achieved by classically equivalent but constructively distinct descriptions of the underlying set.

This approach allows a direct reuse of theorems of constructive mathematics for computable analysis and permits coding-free proofs of results in TTE.

A Step towards a Complexity Theory for Dynamical Systems

KLAUS MEER
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(joint work with MARCO GORI)

We study dynamical systems as continuous time computational devices. Our goal is to introduce complexity classes and notions of reducibility and completeness for particular such systems. The idea is to follow a Lyapunov approach relating a continuous problem to energy functions and their global minimizers. We introduce two classes U and NU of problems characterized by the structure of local minima of related energy families. Some completeness results are established.

The Sets Fixable by Computable Maps on the n-Ball

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We say that $X \subseteq I^n$ is *fixable* if it is the set of points fixed by some computable function $f : I^n \to I^n$. It should be clear that a fixable set must be Π^0_1 and, by Brouwer's theorem, non-empty. But are all non-empty, Π^0_1 subsets of I^n fixable? The following theorem answers this question in the negative and classifies the fixable subsets of I^n.

Theorem 1.
Let $X \subseteq I^n$ be Π^0_1. The following are equivalent:

1. X is fixable
2. X contains a nonempty, connected, Π^0_1 subset.
3. For all computable $f: X \to \mathbb{R}$, $f[X]$ contains a computable real.

This theorem generalizes Orevkov's 1963 constructive counterexample to Brouwer's theorem.

Fine-Computable Functions and Their Walsh-Fourier Coefficients

Takakazu Mori
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For a Fine-computable sequence of functions, we introduce a notion of effective Fine-convergence. If f is Fine-computable, then there exists a computable sequence of binary step functions, which Fine-converges to f. On the other hand, there exists a computable sequence of binary step functions, which Fine-converges to a non-computable function. We also define effective integrability for Fine-computable functions. We prove that the Walsh-Fourier coefficients of an effectively integrable Fine-computable function form a computable sequence of reals and converges effectively to zero.

Real Numbers and BDDs

Norbert Th. Müller
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Quite often, applications from practice as well as rather 'theoretical' representations of real functions in TTE require storing large amounts of data. So in practice, the main memory of computers turns out to be the most important bottleneck at the time, while in complexity theory exponential lower bounds result from the usual represenations of functions.

We discuss the idea of using binary decision diagrams (BDDs) to store finite precision approximations to real numbers in order to improve the existing approaches.

On the Computational Complexity of Best L_1-Approximation

Paulo Oliva
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We present an upper bound on the complexity of the sequence $(p_n)_{n\in\mathbb{N}}$ of best L_1-approximations of a polynomial-time computable function on the interval $[0, 1]$ from the space of polynomials of bounded degree P_n. The analysis makes essential use of the modulus of uniqueness for L_1-approximation presented in KO'01 [U. Kohlenbach and P. Oliva. Proof mining in L_1-approximation. (35 pages), 2001, submitted].

Effectivity in Admissibly Representable Spaces

Matthias Schröder
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A multirepresentation δ_X of a set X is a surjective correspondence between the Baire space $\mathbb{N}^{\mathbb{N}}$ and X. A function $f :\subseteq X \to Y$ is called relatively computable (continuously realizable) with respect to multirepresentations δ_X and δ_Y, iff there is a computable (continuous) function on the Baire space transforming every name of an argument x into a name of the corresponding result $f(x)$. Relatively computable functions as well as continuously realizable functions turn out to be sequentially continuous ones between the limit spaces induced by the used multirepresentations. The property of *admissibility* is defined in such a way that continuous realizability w.r.t. admissible multirepresentations is equivalent to sequential continuity.

We introduce an operator $*$ that transforms every multirepresentation δ into an admissible multirepresentation of the limit space induced by δ. This operator does not only preserve continuous realizability, but also relative computability, i.e. every function $f :\subseteq X \to Y$ which is computable w.r.t. multirepresentations δ_X and δ_Y is also computable w.r.t. δ_X^* and δ_Y^*. We also define an operator which maps δ to an admissible multirepresentation of the topological space that is equipped with the final topology of δ.

Compactness and Continuity Revisited

Peter M. Schuster
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(joint work with Douglas Bridges and Hajime Ishihara)

The relationships between various classical compactness properties, including the constructively acceptable one of total boundedness and completeness, are examined using intuitionistic logic. Even the Bolzano–Weierstraß principle, that every sequence in a compact metric space has a convergent subsequence, is brought under our scrutiny; although that principle is essentially nonconstructive, we produce a reasonable, classically equivalent modification of it that is constructively valid.

Relating Functional Paradigms for Exact Real Arithmetic

Alex Simpson
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(joint work with Andrej Bauer, Martin Escardo)

In this talk, we investigate and compare two type hierarchies of total functionals over the real numbers. The first type hierarchy, which we call the "extensional" hierarchy arises independently as the hierarchy of total functionals induced in each of the following settings:

E1. Scott-continuous functionals over the continuous interval domain

E2. the type hierarchy over $\mathbb{R}$ in the cartesian-closed category of sequential topological spaces

E3. the type hierarchy over $\mathbb{R}$ induced by "extended admissible" representations of functionals over $\mathbb{R}$ in the sense of Schroeder

E4. the total functionals computable in RealPCF (relative to oracles for arbitrary functions from $\mathbb{N}$ to $\mathbb{N}$)

The second "intensional" type hierarchy arises in either of the following two settings:

I1. Scott-continuous functionals over the algebraic domain of signed-binary representations of real numbers.

I2. the total functionals over $\mathbb{R}$ computable in PCF++ (relative to oracles for arbitrary functions from N to N) using a signed-binary representation of $\mathbb{R}$

Our two main results concern types of second-order and below, where the quintessential second-order type is $(\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$ (the type of the definite-integral operator).

THEOREM 1. For all types of second-order and below, the intensional hierarchy and extensional hierarchies coincide.

THEOREM 2. For all types of second-order and below, a third equivalent "intensional" characterisation is:

I3. the total functionals over $\mathbb{R}$ computable in PCF (relative to oracles for arbitrary functions from $\mathbb{N}$ to $\mathbb{N}$) using a signed-binary representation of $\mathbb{R}$

I.e. the parallel features of PCF++ are dispensable. This theorem is proved by reducing to the third-order case of Dag Normann's result that the parallel features of PCF are not required for programming total functionals over $\mathbb{N}$.

We know little about what happens at higher types. However, the situation at third-order is closely related to the answer to the following question. Let $\mathbb{N}$ be the discrete space of natural numbers, and let $\mathbb{B}$ be Baire space (i.e. $\mathbb{N}^\omega$ with the product topology). Let $\mathbb{N}^\mathbb{B}$ be the set of continuous functions from $\mathbb{B}$ to $\mathbb{N}$ topologised with the topology of sequence convergence.

QUESTION Is $\mathbb{N}^\mathbb{B}$ zero-dimensional?

A positive answer to this question will allow a central difficulty in extending Theorem 1 to third order to be surmounted.

Well Computable Real Numbers

Dimiter Skordev
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A real number is computable iff there is an algorithm for constructing arbitrarily close rational approximations of it. No restriction on the complexity of the algorithm is imposed. However, the concrete real numbers arising in naturally formulated problems of analysis usually allow their approximations to be constructed by algorithms with not very high complexity.

It is reasonable to look for some restricted class of algorithms that is sufficient for such needs of analysis. We give certain arguments in favour of the algorithms corresponding to primitive recursive functions. It is quite possible that some even more restricted classes of algorithms can also turn out to be appropriate.

Unbounded operators on Hilbert spaces in constructive mathematics

Bas Spitters
University of Nijmegen, Netherlands

We consider the question: 'Can constructive mathematics be applied to physics?'. More specifically: 'Is there a good constructive theory of unbounded operators?'. To answer this question I will discuss operators with located graph. I will show that a bounded operator has an adjoint if and only if its graph is located. Locatedness of the graph is a necessary and sufficient condition for an unbounded normal operator to have a spectral decomposition. These results suggest that located operators are the right generalization of bounded operators with an adjoint.

Properties of Real Interpretations of ω-Words

Ludwig Staiger
Martin-Luther-Universität Halle-Wittenberg, Germany

We consider ω-words ξ as r-adic expansions $\nu(\xi)$ of points in the d-dimensional unit cube $[0,1]^d$. Our aim is to define subsets of $[0,1]^d$ by subrecursive devices. This yields a connection between Computable Analysis and the Theory of Formal Languages. For our topological considerations we use the usual topology of the unit cube and we consider the set of all ω-words over a fixed finite alphabet X as the Cantor space X^ω.

In the talk we confine to the case of finite automata. Sets of ω-words defined by finite automata (so-called regular ω-languages) are widely investigated an enjoy nice properties:

So the set of regular ω-languages is a Boolean Algebra and further closed under taking the closure of a set. Moreover the ω-language $\nu^{-1}(\nu(F))$ containing all preimages $\nu^{-1}(\nu(\xi))$ of the image $\nu(\xi)$ of an ω-word $\xi \in F$ is also regular provided F is regular.

A second property is the following measure-category-theorem (cf. [1]): Let be given a nonvanishing balanced measure on X^ω. Then every regular $F \subseteq X^\omega$ is a nullset iff it is of first Baire category in the Cantor space X^ω.

These two properties carry over to the family of sets $\{\nu(F) : F \subseteq X^\omega \wedge F \text{ is regular}\}$.

It is mentioned that the family of sets $\{\nu(F) : F \subseteq X^\omega \wedge F \text{ is regular}\}$ as well as the balance condition for a measure on X^ω induced by a measure m on the unit cube depend on the base r chosen. Moreover, the measure-category-theorem is no longer true for the slightly more complicated one-counter ω-languages.

[1] Staiger, L. (1998), Rich ω-words and monadic second-order arithmetic, in: Computer Science Logic, 11th International Workshop, CSL'97, Selected Papers (M. Nielsen and W. Thomas Eds.), Lecture Notes in Comput. Sci. No. 1414, Springer-Verlag, Berlin. pp. 478 – 490.

Representations of Functionals

Takeuti Izumi
Kyoto University, Japan

It is important to compute a functional over real numbers, which is a function of functions of reals, in the real computation, especially when one solves differential equations or integral equations. One has to give representations to functionals over real numbers in order to discuss computability and complexity for computation of the functionals. This work shows that for each functional over real numbers, there exists a representation which is a functional over natural numbers.

A Domain-Theoretic Account of Computability via Bottomed Sequences

Hideki Tsuiki
Kyoto University, Japan

The space $\Sigma^\omega_{\perp,n}$ of bottomed sequences, which are infinite sequences in which n undefined cells are allowed to exist, is used to define computation over topological spaces. Since a bottomed sequence is a kind of sequence, we can define a machine called an IM2-machine which input/output sequences in $\Sigma^\omega_{\perp,n}$ with $n+1$ heads and indeterministic rules. At the same time, since $\Sigma^\omega_{\perp,n}$ is a weak topological space, we can embed n-dimensional separable metric spaces into $\Sigma^\omega_{\perp,n}$ as subspaces. Therefore, we can consider that an IM2-machine operates directly on the space elements via the embedding. From this fact, we expect that some of the properties of the space is reflected on the machine structure. The coincidence of the topological dimension of the space and the number of extra heads required to make computation is one example.

In this talk, we present the domain-theoretic aspect of computation by IM2-machines. We introduce the notion of a finitely generated domain, which is an algebraic domain with a well-founded finite branching base and with labeling of the edges by an alphabet. The set of finite/infinite states of an I/O tape of an IM2-machine corresponds to compact/limit elements of a finitely generated domain, and therefore we can naturally consider that an IM2-machine operates on a finitely generated domain. It is also shown that an n-dimensional compact separable metric space is homeomorphic to the set of minimum elements of the limit space of a n-dimensional finitely generated domain, which corresponds to proper representation in TTE theory.

Computational Complexity on Computable Metric Spaces

Klaus Weihrauch
FernUniversiät Hagen, Germany

We introduce a new Turing machine based concept of time complexity for functions on computable metric spaces. It generalizes the ordinary complexity of word functions and the complexity of real functions studied e.g. by Ko. Although this definition of TIME as the maximum of a generally infinite family of natural numbers looks straightforward, at first glance, examples for which this maximum exists seem to be very rare. It can be shown that, nevertheless, the definition has a large number of important applications. Using the framework of TTE, we introduce computable metric spaces and computability on the compact subsets. We prove that every computable metric space has a *c-proper* c-admissible representation. We prove that Turing machine time complexity of a function computable relative to c-admissible c-proper representations has a computable bound on every computable compact subset. We prove that computably compact computable metric spaces have *concise* c-proper c-admissible representations and show by examples that many canonical representations are of this kind. Finally, we compare our definition with a similar but not equivalent one by Labhalla et al. By these results natural and realistic definitions of computational complexity are now available for a variety of numerical problems such as image processing, integration, continuous Fourier transform or wave propagation.

Metrization of the Uniform Space and Effective Convergence

Mariko Yasugi
Kyoto-Sangyo University, Japan
(joint work with Y. Tsujii, T. Mori)

It is a classical fact that the countable uniform space can be metrized. There is a general construction of a metric from the uniformity. On the other hand, we can define the notion of effective convergence in the effective uniform space. We can then show the equivalence of effective convergence with respect to the uniform topology and effective convergence with respect to the induced metric (without knowing if the metric is computable).

Computability and Entropy Solutions of Conservation Laws

Atsushi Yoshikawa
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A typical example of systems of conservation laws is the inviscid Burgers equation $u_t + (u^2/2)_x = 0$ for $u = u(t, x)$. Even started from the smooth initial data $u(0, x)$, its solution generally loses smoothness as the time elapses as well as uniqueness. It means that mere generalized solutions will not suffice to handle such equations. However, the class of entropy solutions makes sense globally and also sustains uniqueness. There generally are certain algorithmic machineries, relating the initial values to the entropy solutions, of which, in the case of the inviscid Burgers equation, the Cole-Hopf transformation is typical. Although the validity of such machineries is verified in the realm of mathematical analysis, their very nature should be analyzed as a computability question. Thus, if the initial data is computable (in $\mathbf{L}^1$, in the sense of Pour-El and Richards, for instance), then one would like to ask how the entropy solution is together with effectivity of the machineries. As for the inviscid Burgers equation, this is supported by the classical discussion of Hopf (Comm. Pure Appl. Math., 3(1950)), which leads readily computable interpretation. There are evidences that this speculation might be extended to a much broader class of conservation laws, as recent works of Bressan (Hyperbolic systems of conservation laws, Oxford University Press, 2000) and Kohlenbach's proof mining suggest.

Beyond the Computable Real Numbers

Xizhong Zheng
BTU Cottbus, Germany

A real number x is computable if it is a limit of some computable sequence of rational numbers which converges to x effectively. If the condition of "effective convergence" is replaced by other weaker conditions, the superclasses of computable real numbers can be defined. In this talk we summarize several recent results about such classes of real numbers. Among others, the properties about the classes of left computable, semi-computable, monotonically computable, weakly computable, divergence-degree-bounded computable, recursively approximable, and more generally the arithmetical real numbers are discussed.

Computable Analysis of Partial Differential Equations

Ning Zhong
Clermont College, University of Cincinnati, USA
(joint work with Klaus Weihrauch)

In this talk we discuss computability of the solution operators of several partial differential equations, including wave equation, linear and nonlinear Schrödinger equations, and the KdV equation. An affirmative answer is given to one of the open problems posted by Pour-El and Richards: Whether the solution operator of the KdV equation is computable?

Computability on Regular Subsets of Euclidean Space

Martin Ziegler
Heinz Nixdorf Institute, University of Paderborn, Germany

Algorithms in Computational Geometry rather frequently suffer, upon implementation on actual digital computers, from severe numerical instabilities. We find that the theory of computable real number functions can provide much insight into the cause of such problems. In fact, several works in Recursive Analysis already dealt with geometric problems such as convex hull and the like. However, each author seems to have introduced his own notion of a computable subset of Euclidean space.

We systematically compare these notions, showing that each one is computationally equivalent to one pair out of 8 'basic' representations for so-called *regular* sets: 4 ones encoding 'positive' and 4 ones for 'negative' information. The correspondingly induced notions of computability turn out to be different in general but, for convex sets, they coincide.