Mathematical Structures for Computable Topology and Geometry

26.05 - 31.05.2002

organized by

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Topological notions and methods have successfully been applied in various areas of computer science. Computerized geometrical constructions have many applications in engineering. The seminar we propose will concentrate on mathematical structures underlying both computable topology and geometry.

Due to the digital nature of most applications in computer science these structures have to be different from the mathematical structures which are classically used in applications of topology and geometry in physics and engineering and which are based on the continuum. The new areas of digital topology and digital geometry take into account that in computer applications we have to deal with discrete sets of pixels.

A further aspect in which topological structures used in computer science differ from the classical ones is partiality. Classical spaces contain only the ideal elements that are the result of a computation (approximation) process. Since we want to reason on such processes in a formal (automated) way the structures also have to contain the partial (and finite) objects appearing during a computation. Only these finite objects can be observed in finite time.

At least three types of computationally convenient structures for topology have been studied, and all of them may be developed in the direction of geometry. The first is domains, the second locales (and formal topology), and the third cell complexes.

Domains, originally introduced by Dana Scott for the formal definition of programming language semantics, have recently found a broader field of applications. Domain theory provides interesting possibilities for exact infinitary computation. There are the “maximal point models”. The interval domain in which the real numbers are embedded as maximal elements is an example of this. Escardó has used it for his development of a programming language that allows computing with intervals. But there are also domain models for convexity and intended applications in computer-aided design (Edalat et al.).

Closely related to domain theory is the theory of locales (Coquand, Resende, Vickers, ...) and its logical counterpart: formal topology (Martin-Löf, Sambin,

. . .). Here, one takes a constructive attitude and starts from the already mentioned fact that only the finitely describable properties of the ideal mathematical entities we are interested in are observable. Thus, these properties are the primary object of study. The ideal entities are obtained as derived objects. Formal topology is an open system based on Martin-Löf's type theory that allows the derivation of topological and (at the moment only to a certain extent) geometrical statements.

In geometry a similar approach, bypassing mainstream mathematics, tries to develop a system suitable for "commonsense" spatial reasoning, by taking regions rather than points as the basic entities. Developed initially by philosophers under the name mereology (and later, mereotopology), this viewpoint has been taken up by researchers interested in applications in AI, robotics, and GIS. The best-known product of this recent work in computer science is the so-called RCC (region connection calculus), also named for its originators Randell, Cohn and Cui.

This region-based topology/geometry has remained rather isolated from those mathematical disciplines which might be expected to interact fruitfully with it. In particular, there is an obvious analogy with point-free topology (as above), but there has been relatively little interaction so far. (It is one intention of the seminar to help changing this.) Again, the region-based theories have typically had the infinite divisibility of space built in; attempts at a discrete version are few. Here one may expect that cell complex theory in which, after all, the cells are usually thought of as convex regions could help.

Combinatorial topology offers us discrete (or finitary) structures which have long played a part in image processing: cell complexes. These may be either "concrete" (derived explicitly from Euclidean space, or more generally from a manifold), or "abstract". The concrete complexes do not provide us with the autonomous theory we are looking for; the abstract complexes permit the computation of various topological invariants, but do not support specifically geometric features such as convexity and linearity. To make progress, it seems that we need either to endow the abstract complexes with suitable extra structure, or else to ground them in some richer combinatorial structures (not classical manifolds).

In the latter connection, it is worth mentioning oriented matroids. Despite pioneering work by Knuth (1991), these have been almost completely ignored by computer scientists. Briefly, a matroid can be described as a simplicial complex with just enough extra structure to handle linear dependence. An oriented matroid then admits just enough further structure so that one can deal with convexity as well as linearity. It seems likely that oriented matroid theory will have a significant input to the (eventual) foundations of digital geometry, even if this has been little recognized so far.

The aim of the workshop was to bring together people working in fields like domain theory, computer science oriented topology and geometry, formal topology,

. . . and to foster interaction between them. 57 top scientists and promising young researchers accepted the invitation to participate in the challenging experience. They came from 16 countries, mostly European countries and the USA, but also China, Japan, Mexico, New Zealand, Russia, South Africa and Turkey. The 45 talks covered all of the areas mentioned above.

The workshop was a great success. Many new cooperations were started. The particpants expressed high appreciation of this gathering and praised the extraordinary Dagstuhl.

As organizers of the Dagstuhl seminar on Mathematical Structures in Computable Topology and Geometry and on behalf of the particpants we want to thank the institute and its staff, both in Saarbrücken and in Dagstuhl, for the excellent work they did to make it all run smoothly in an efficient but always pleasant and friendly manner.

Contents

1 Combinatorial Differential Manifolds and Matroid Bundles

Laura Anderson

We survey a combinatorial analog to the theory of differential manifolds and real vector bundles, in which the role of real vector spaces and coordinate charts is played by oriented matroids. Work by several people, culminating in a recent result by Daniel Biss, shows that this model for vector bundles is equivalent to the theory of real vector bundles over triangulable base spaces.

2 Bar Recursion on the Continuous Functionals

Ulrich Berger[1]

Bar recursion is, roughly speaking, recursion on the well-founded tree defined by a continuous functional $Y\colon (\mathbb{N}^{\mathbb{N}} \to \rho) \to \mathbb{N}^{\mathbb{N}}$, where ρ is a topological space. Bar recursion and related forms of recursion have been used in proof theory to provide a constructive interpretation of various choice principles (countable choice, dependent choice) within nonconstructive theories. In this talk, we discuss the relationship between the following three forms of bar recursion:

Spector (SBR) $$\Phi(s) \overset{\tau}{=} \begin{cases} G(s) & \text{if } Y(s \mathbin{@} \lambda k.0^\rho) < |s| \\ H(s, \lambda x^\rho.\Phi(s * x)) & \text{otherwise.} \end{cases}$$

Kohlenbach (KBR) $$\Phi(s) \overset{\tau}{=} \begin{cases} G(s) & \text{if } Y(s \mathbin{@} \lambda k.0^\rho) \overset{\rho}{=} Y(s \mathbin{@} \lambda k.1^\rho) \\ H(s, \lambda x^\rho.\Phi(s * x)) & \text{otherwise.} \end{cases}$$

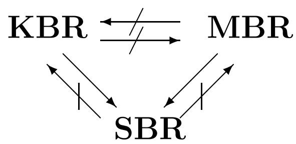
Modified bar recursion (MBR) $\Phi(s) = Y(s \mathbin{@} \lambda k.H(s, \lambda x^\rho.\Phi(s * x)))$

where $s \in \rho^*$, $G\colon \rho^* \to \mathbb{N}^{\mathbb{N}}$, $Y\colon (\mathbb{N}^{\mathbb{N}} \to \rho) \to \mathbb{N}^{\mathbb{N}}$, $H\colon \rho^* \times (\rho \to \mathbb{N}^{\mathbb{N}}) \to \mathbb{N}^{\mathbb{N}}$, all continuous, $(s \mathbin{@} \alpha)(k) := s_k$, if $k < |s|$ and $:= \alpha(k)$, otherwise. Setting

$$\Phi \longrightarrow \Psi :\equiv \Phi + \text{ Gödel primitive recursion defines } \Psi$$

the main results can be summarized by the following diagram:

[1]joint work with Paulo Oliva; Ulrich Berger is funded by the British EPSRC and Paulo Oliva by the Danish National Research Foundation



3 A Topological Representation Theorem for Oriented Matroids

Jürgen Bokowski

The talk mentioned a new direct proof of a topological representation theorem for oriented matroids (joint work with Simon King, Susanne Mock, and Ileana Streinu). It uses hyperline sequences and the generalized Schönflies theorem.

For *mathematicians, computer scientists and engineers* matrices have become so useful in all its applications that they hesitate very much to give up this concept in order to generalize it to that of oriented matroids. But there are many reasons to do so, especially when the matrices are used to describe *geometric objects* such as polyhedra, convex polytopes, hyperplane arrangements, or finite point sets, etc., and the investigation concerns *combinatorial properties* of these geometric objects.

An *oriented matroid* is a generalized rigid motion invariant *matrix function of a high geometrical relevance.* More than twenty years of an exciting development in mathematics together with early contributions since 1926 have created a *theory of oriented matroids* the fundamentals of which deserve to be known by mathematicians, computer scientists and engineers because of its applications and simplications.

Oriented matroids turn out to be even an invariant under topological transformations of the projective space which then require a definition leaving the framework of Euclidean geometry. But instead of leading to a drawback, this fact has a decisive advantage: the *invariant of the projective space* can always be described as a *finite sequence of signs* and this in turn allows *computations* in terms of simple sign considerations.

There are many completely different definitions for an oriented matroid that do not tell the novice really what this structure is about. The talk not even mentioned the three equivalent definitions in detail that the above article is about: hyperline sequences, chirotopes, and sphere systems.

4 Computable Versions of the Graph Theorem

Vasco Brattka

The Graph Theorem states that a function is computable, if and only if its graph is recursive. We investigate up to which extend this theorem can be generalized to computable metric spaces. Besides the stated non-uniform version we also study the uniform question whether the transformation from the function to the graph and its inverse are computable. It turns out that the transformation from the function to its graph is computable in general, while the inverse transformation from the graph to the function fails to be computable in certain cases. A counterexample shows that this even happens in the case of linear operators on Hilbert spaces. On the other hand, our main result shows that for the Euclidean space as target space, it is sufficient that the source space is effectively locally connected in order to obtain a computable transformation from the graph to the function.

5 Textures and Ditopology

Lawrence Michael Brown

A texturing on a set S is a point separating, complete, completely distributive lattice $\mathcal{S}$ of subsets of S containing S and $\emptyset$, and for which meets coincide with intersections and finite joins with unions. The pair $(S, \mathcal{S})$ is then known as a texture. Textures form an ideal setting for the development of complement-free mathematical concepts, as well as providing a set-based framework for the study of fuzzy sets.

The correct notion of topology in the context of textures is that of a ditopology, in which the open sets and closed sets are given independently of one another. Two important constructs, namely those of dicover and direlation, are discussed in this talk. Dicovers allow a powerful characterization of the notion of dicompactness for ditopological texture spaces. In addition it is shown that the classical connection between covers and binary relations generalizes to dicovers and direlations, permitting two equivalent characterizations of a notion of uniformity on textures.

Finally it is noted that the concept of difunction between textures, which is derived from that of direlation, has many of the desirable properties of classical functions between sets. It is therefore natural to consider the category of textures and difunctions, and also that of ditopological texture spaces and bicontinuous difunctions.

Some current work in the above areas is mentioned in the talk.

6 Compactly Generated Regular Locales

Martin Escardo

A space is exponentiable in the category of topological spaces if and only if it is core-compact (Day and Kelly 1970) and thus the category of topological spaces is not cartesian closed. However, it has various full subcategories which are. They include the sequential spaces, the compactly generated Hausdorff spaces (going back to Kelley 1950's), and the quotients of exponentiable spaces (Day 1972), among others. This work is an attempt to identify full subcategories of the category of locales which are cartesian closed. We define the Kelleyification of a Hausdorff locale to be the colimit of the diagram whose vertices are its compact sublocales and whose arrows are the existing inclusions between the vertices, and we say that the locale is compactly generated if it is canonically homeomorphic to its Kelleyification. The main result is that the topology the Kelleyification of a regular locale is isomorphic to the second Lawson dual of the topology of the locale (most of the lemmas toward this result hold more generally for Hausdorff locales). In particular, for regular locales, Kelleyification is functorial making the canonical map into a natural transformation. Another consequence is that compactly generated regular locales have enough compact sublocales to separate the opens (and hence, classically, have enough points). A sufficient condition for obtaining a cartesian closed category of compactly generated locales is that Kelleyification be a coreflection. Some steps in this direction have been discussed; the main difficulty is to identity separation properties that are preserved by the construction.

7 Topology via the Lambda-Calculus: Ten Theorems in General Topology whose Proofs Together Fit in the Margin of a Single Page of Fermat's Book

Martin Escardo

The central notion in topology is that of continuity. With the aid of the so-called Sierpinski space, many other notions can be reduced to it, including those of openness, closedness, Hausdorffness and compactness. The general idea is that a space is compact if and only if a certain function is continuous, that it is Hausdorff if another function is continuous, and so on. Using the fact that compositions of continuous functions are themselves continuous, one gets some theorems "for free". However, function composition is not quite enough to tackle theorems of interest. An algebra of functions, known as the lambda-calculus, which generalizes function composition, allows us to prove many interesting theorems in a brief and transparent manner. The basic lemma says that every function which is definable from continuous functions using lambda-algebra is itself continuous. This is exemplified by ten theorems in general topology, ranging from relatively easy to moderately hard, which admit such algebraic proofs. The proofs are not only extremely short, but they also provide additional insight, regarding both why the theorems are true and what they actually say. Some arguments appear to be magic, such as a half-line proof of the binary Tychonoff theorem. However, one can argue, the proofs are the same as the classical ones; what is going on is that the lambda-calculus machinery is performing and hiding all the routine bookkeeping, freeing us to perform and see only the relevant, interesting steps.

8 A Topological Approach to the Game of "20 Questions"

Robin Forman

We will show that a variety of techniques from differential topology, the study of smooth manifolds, can be adapted to provide tools for the study of combinatorial spaces. Moreover, such ideas can be applied to the study of some questions in

the theories of computability and complexity. As an example, we show that a combinatorial version of Morse theory, a fundamental tool in differential topology, provides a topological point of view from which one can examine a very general form of the game of "20 Questions".

9 4D Adjacency Relationships and Thinning

Chyi-jou Gau[2]

The n-xel of a grid point p in an n-dimensional Cartesian grid is p's Voronoi neighborhood. A 4-xel is an "upright" 4-dimensional hypercube. Two 4-xels are said to be 80-,64-,32- or 8-adjacent if they are distinct and their intersection is a k-xel for $k \geq 0, 1, 2,$ or 3 respectively. A 4D binary image is a finite set of 4-xels. A simple point q in a binary image G has the property that its deletion "preserves topology". This property depends on the adjacency relationships of q with its adjacent 4-xels in G.

It is well known in digital topology that a specified parallel thinning algorithm always "preserves topology" of the binary image G if no iteration of that algorithm can ever delete a minimal non-simple ("MNS") set of image G. An easily visualized characterization of simple points on 3D and 4D Cartesian grid (and other grid systems), based on their attachment sets, was introduced by the second author. We use the same general approach to determine which set of 4-D xels can be MNS, and also determine which of those sets can be MNS without being a component of the image G. This provides us a sufficient condition to ensure that a parallel thinning algorithm always "preserves topology" in 4D digital space.

We can "stack" a sequence of 3D images of a moving object to form a 4D image. When this 4D image is thinned, the resulting skeleton can provide information on the motion of this object. We believe that this is a useful first step to analyze the continuous motion of 3D objects.

Keywords: 4-xel, simple, attachment set, minimal non-simple (MNS) set, parallel thinning, topology preservation.

[2]joint work with T. Yung Kong

10 The Basic Picture as Invariance under Transfer along a Relation

Silvia Gebellato[3]

The topological notions of open and closed subsets, which are usually presented resorting to a notion of nearness, have been characterized mathematically in terms of existential-universal and universal-existential images of subsets along a relation between two sets, see [4]. This shows that open and closed are logically dual. Moreover, one can see that a geometrical symmetry links the pointfree with the pointwise notions.

In a similar way, the notion of continuous map has been characterized in a structural way in terms of commutative squares, see [1], and a pointfree (or formal) notion of continuity has been obtained just by symmetry from the pointwise one, see [2].

This new and general approach to topological notions has been called *the basic picture*. Formal topology, as in [3], is obtained simply by adding a requirement of convergence.

Now our aim is to find a fully structural, self-contained explanation of all the definitions of the basic picture. The results described above seem to support our claim that the basic picture is what remains invariant under transfer along a relation. A few results in this direction have been presented.

[1] S. Gebellato and G. Sambin, *The essence of continuity (the Basic Picture, II)*, Preprint n. 27, Dipartimento di Matematica P. e A., Università di Padova, 2001.

[2] S. Gebellato and G. Sambin, *Pointfree continuity and convergence (the Basic Picture, IV)*, first draft, 2001.

[3] G. Sambin, *Intuitionistic formal spaces - a first communication*, in Mathematical Logic and its Applications, ed. D. Skordev, Plenum, 1987, pp. 187-204.

[4] G. Sambin, *The Basic Picture, a structure for topology (the Basic Picture, I)*, Preprint n. 26, Dipartimento di Matematica P. e A., Università di Padova, 2001.

[3]joint work with Giovanni Sambin

11 Sigma-Frames, Domains and Dualities

Christopher Gilmour

A sigma-frame is a bounded lattice in which each countable subset has a join and for which meet distributes over countable joins. Those familiar with frame theory will be comfortable with most of the constructions for sigma-frames, but there are some nice surprises which especially shed light on the spatial counterparts. The purpose of this talk is to present some of the background to sigma-frames and the associated dualities, to present a few new results which hopefully sit within the declared purpose of the workshop , and, in turn, to be informed by the workshop on possible motivations from computer science.

12 Domain Representations of Convergence Spaces

Reinhold Heckmann

Convergence spaces are the objects of a cartesian closed category CONV that includes the category TOP of topological spaces, which is not cartesian closed. They are usually described by saying which filters converge to which points. Here we present a description of CONV without filters, namely a category of domain representations that is equivalent to CONV. Its objects are usually smaller and more manageable than the collection of filters of a set. The usual embedding of TOP into CONV is translated into this new setting, leading to omega-algebraic representations for second-countable spaces. We also show how to construct domain representations for products, disjoint sums (coproducts), subspaces, function spaces, and quotients from domain representations of the operand spaces.

13 (Default) Negation for Logic Programming in Algebraic Domains

Pascal Hitzler

Recently, William C. Rounds and Guo-Qiang Zhang have proposed notions of resolution and logic programming in a clausal logic on algebraic domains. We explore their paradigm by studying two classical issues from logic programming and nonmonotonic reasoning.

Firstly, we describe conditions on the underlying algebraic domain which allow to obtain a resolution theorem stating that a clause is a logical consequence of a theory if and only if the empty clause is derivable from the theory together with the negated clause. Such a form of resolution theorem is fundamental for the theory of classical logic programming. The conditions which we impose on the domain in order to obtain this theorem encompass a condition related to dI-domains and coherence spaces, and a form of negation.

Secondly, we enhance the logic programming paradigm due to Rounds and Zhang by a notion of default negation, borrowed from nonmonotonic reasoning. We will see that fundamental results concerning the answer set semantics can be carried over.

References

P. Hitzler, Resolution and Logic Programming in Algebraic Domains: Negation and Defaults. Technical Report WV-02-05, Knowledge Representation and Reasoning Group, Department of Computer Science, Dresden University of Technology, 2002.

W.C. Rounds and G.-Q. Zhang, Clausal Logic and Logic Programming in Algebraic Domains, Information and Computation 171(2) (2001) 156-182.

14 Polyhedral Complexes for Computer Imagery: 3D Border Tracking with Surface Structures

Yukiko Kenmochi[4]

There already exist various border tracking algorithms of objects in a 2D/3D digital image. One of basic and useful ideas to solve the border tracking problem in 2D is to use a curve structure of the border of a 2D object. The simple extension of the border tracking in 3D will be then to use a surface structure of the border of a 3D object. However, it is actually difficult to find a good

[4]joint work with Atsushi Imiya

definition or representation of 2D surfaces in a 3D discrete space such as a 3D digital image.

In the talk, we first give the overview of 3D border tracking problems and classification of various approaches for defining 2D discrete surfaces. We then use some concepts of polyhedral complexes and their boundaries in combinatorial topology and algebraic topology for providing surface structures in a 3D discrete space. The similar concepts are already used in some of previous work (but in different ways). We present an algorithm for tracking the combinatorial boundary from a given 3D digital image. We also show that our combinatorial boundary becomes a triangulation of border points in the sense of general topology, obtained by a set operation by using a neighborhood.

15 Digital Geometry for Image Analysis - Length and Surface Area

Reinhard Klette

Multigrid convergence studies feature estimators under the assumption of decreasing grid constants. Digitization models as used by Gauss or Jordan are used to relate unknown Euclidean sets to digitized objects in image analysis.

Digital straight segment approximations, minimum-length polygon calculations and integrations of approximated normals are known methods providing multigrid convergent length estimation for convex curves with linear speed of convergence. Based on sampling of curves it is possible to achieve faster convergence speed. The open question remains: are there superlinear convergence speed length estimators for digitized curves?

Digital surface segment approximation possesses experimentally measured multigrid-convergence behavior, but no convergence theorem and resulting ‘surface patches’ do not form a simple polyhedron even in case of digitized convex solids. Integrations of approximated surface normals possess a convergence theorem, a (not very efficient) implementation, and may fail to calculate simple polyhedra if a simple isothetic polyhedra is used as input. Relative convex hulls possess a convergence theorem, calculate simple polyhedra for Jordan digitizations being simple isothetic polyhedra for both inner and outer digital sets, but no polygonal speed algorithm is known for this method,.

16 On Finite Approximation of Compact Hausdorff Spaces

Ralph Kopperman

Each compact Hausdorff space is the Hausdorff reflection of the inverse limit of a system of finite T_0-spaces and continuous maps. It has recently been learned that these continuous maps can be assumed to also be *chaining:* the image of the closure of each element is a specialization chain (that is, given any two elements of this image, one is in the closure of the other).

When the maps are chaining, the inverse limit and all its skew compact subspaces are normal topological spaces (though rarely Hausdorff). Further, each point is above a unique closed point, and the subspace of closed points together with the map which assignes to each point this closed point in its closure, is the Hausdorff reflection of the space.

A *(finite) simplicial complex* is a finite set V of points in general position in some Euclidean space (called its *vertices*) and a set of nonempty subsets of V (called *simplices*), K such that if $\sigma \in K$ and $\emptyset \neq \rho \subseteq \sigma$ then $\rho \in K$. We view a finite simplicial complex K as a T_0 space with the Alexandroff topology whose open sets are the $\subseteq$-upper sets.

The *polytope* $|K|$ of a simplicial complex K, is union of the convex hulls of its simplices; and given polytopes $|K|, |L|$, a *simplicial* map $|f| : |K| \to |L|$ is one such that $|f|$ takes vertices in a common simplex of K to vertices in a common simplex of L, and is defined linearly on simplices. For continuous map between simplicial complexes, $f : K \to L$, its associated simplicial map $|f| : |K| \to |L|$ is defined by $|f|(\Sigma r_v v) = \Sigma r_v f(v)$, and given such a simplicial map, its associated continuous map $f : K \to L$ is defined by $f(\sigma) = |f|[\sigma]$ (the simplex whose vertices are the images of the vertices of K under f).

We obtain a representation theorem for Hausdorff reflections: For an inverse sequence of simplicial complexes and chaining maps, the space of minimal points of its inverse limit is homeomorphic to the inverse limit of its associated polytopes and simplicial maps. Conversely, the limit of an inverse sequence of polytopes and simplicial maps is homeomorphic to the space of minimal points of the inverse limit of the sequence of their simplicial complexes and associated continuous maps.

17 Cocompactness and Quasi-Uniformizability of Completely Metrizable Spaces

Hans-Peter A. Künzi

Strengthening a result of Aarts, de Groot and McDowell, we show that a metrizable topological space X is completely metrizable if and only if it admits a quasi-uniformity $\mathcal{U}$ such that the topology induced by the conjugate quasi-uniformity $\mathcal{U}^{-1}$ on X is compact.

Ciesielski, Flagg and Kopperman had asked whether that assertion holds, after establishing an analogous result for separable metrizable spaces in connection with some characterization of computational models.

18 Compactly Generated Spaces: Some Open Problems

Jimmie Lawson

In the category TOP of topological spaces and continuous maps, the exponentiable spaces are known to be those that are core compact, a result dating back to a joint paper of Day and Kelly, 1970. Day later proved (1972) that quotients of these spaces, what we call core compactly generated spaces, form a large cartesian closed subcategory of TOP. We discuss a recent elementary proof of this result and some related results and open questions. For example, it is conjectural, but unknown, whether the core compactly generated spaces form a maximal cartesian closed subcategory of TOP (this question was posed by H. Herrlich). It is also unknown whether there is a core compactly generated space that is not the quotient of a locally compact space.

We also develop ties between the core compactly generated spaces and spaces for which the compact saturated subsets satisfy a dual version of the Hofmann-Mislove theorem. In this setting one would like to have a characterization of those spaces for which every irreducible compact saturated set is the saturation of a singleton set; it is not known whether this holds for all T_0-spaces. The material presented represents joint work with Martin Escardo.

19 Octopus Presentation of 3-Manifolds

Serguei Matveev

An octopus is an oriented graph whose vertices correspond to 3-manifolds and edges to gluing homeomorphisms of their boundary tori. In particular, any 3-manifold can be presented as an octopus consisting of a unique isolated vertex. We describe several moves on octopuses. They make the graphical structure of octopuses more complicated, but vertex manifolds become simpler. In many cases this process converts a given 3-manifold into an octopus with vertices of known types. The next step consists in assembling known vertices to Seifert manifolds. Quite often this integration process terminates with a unique Seifert manifold having an unambiguous name. Computer experiments show that this partial recognition algorithm for 3-manifolds is quite efficient.

20 Measuring the Probabilistic Power Domain

Michael Mislove

In this talk, we describe our work on measurements on the probabilistic power domain. We show how measurements on an underlying domain that satisfy an additional condition naturally extend to the probabilistic power domain so that the kernel of the extension is exactly those measures whose support lies within the kernel of the measurement on the underlying domain. This result is combined with now-standard results from the theory of measurements to obtain a new proof that the limit of a a weakly hyperbolic IFS with probabilities is the unique invariant measure whose support is the attractor of the underlying IFS. This is joint work with Keye Martin (Oxford) and James Worrell (Tulane).

21 Approximated Intersection Versus Exact Topology

Thomas J. Peters

In computer-aided geometric design, geometric models are idealized as compact, regular closed sets containing a finite volume. Model edits are assumed to be the operations of the Boolean algebra of regular closed sets of three dimensional Euclidean space. Yet, computational representations and implementations significantly deviate from these idealized concepts. This is primarily because creation of the boundary of these geometric models relies upon surface intersection algorithms which depend upon approximating numerical methods. This disparity between concept and practice causes serious problems in industry, with annual economic costs exceeding one billion dollars per year. Relevant industrial examples will be shown. A new formalism is needed for these approximated models and editing operations. Some critical criteria and ongoing developments will be discussed.

22 The Faces of Closed Sets

John L. Pfaltz

The usual approach to topologies is through open sets. But in many of the discrete applications we have encountered in computer science, closure seems a more natural concept. In this talk we review an axiomatic theory of closure systems, discuss the notion of the generators of closed sets, and introduce the representation of closure systems by a semi-modular lattice.

Two particular closures are examined in more detail. They are left ideal closure which is uniquely generated, and pseudo convexity which is not. We show that pseudo convexity in graphs is identical to the more familiar node domination concept.

The central topic of the talk then centers on the "faces" of closed sets and their topological intermpretation (if any). A face is the difference between a closed set and a smaller closed set that it covers in the closure lattice.

These closure concepts have been encountered in formal concept analysis, closed set data mining, models of the internet, and image analysis.

23 Computing Triangulations Using Oriented Matroids

Jörg Rambau

TOPCOM is a package for computing triangulations. The input is a point configuration in affine space. A triangulation is a dissection of the convex hull of the configuration into simplices with vertices in the configuration such that any every two simplices intersect in a face of both.

In this expository talk, it is shown how oriented matroids serve as an interface between calculations in the coordinates of the input points and purely combinatorial computations in discrete geometry. The key point is that, in TOPCOM's geometric algorithms, only the relative positions of the points to each other are used, rather than the relative positions of points to a grid. Complete information about the relative positions of the points to each other yields "bad intersection" predicates and "uncovered interior facet" predicates for simplices in a potential triangulation. Computing the so-called chirotope yields all the necessary information: the chirotope of the input configuration is a map that assigns to every affine basis the sign of its determinant (in homogeneous coordinates). This structure represents the complete combinatorial information, the "oriented matroid", of the input configuration, and it allows to compute all triangulations of the input configuration.

24 Topological Antidotes to the State Space Explosion Problem

Martin Raussen

State spaces of parallel processes in concurrency theory can be modelled in a geometric framework as locally partially ordered spaces. Executions correspond to dipaths - repsecting the order - on them. Dihomotopic dipaths represent executions with the same behaviour. Thus it is interesting to study the fundamental category of the space. In order to compress the information contained in it, one introduces the category of fractions with respect to certain natural systems of morphisms. The connected components of the objects with respect to "zig-zag-morphisms" in the system represent the "dicomponents" of the state space in a new component category. The latter is used to shrink the number of states dramatically compared to discrete models.

25 A Paradigm for Program Semantics: Power Structures and Duality

Ingrid Rewitzky

The mathematical notions of power structures and (Stone, Priestley, Jonsson/Tarski) duality provide a paradigm for translating between logical, algebraic and semantic structures. Brink and Rewitzky have shown the applicability of this paradigm for unifying four versions of program semantics - relational model, predicate transformer semantics, information systems and powerdomains. This paradigm is now sufficiently well-established to warrant investigation of further applications. One such application is an analysis of the formal method, called simulation, for determining whether an implementation meets its specification. For this a generalisation of Jonsson/Tarski duality is required to reason about monotone maps that are not necessarily meet- or join-preserving. We present such a duality within the Brink/Rewitzky paradigm for program semantics and using a simple interaction game to motivate the required translations.

26 Ordered Compactifications, Galois Connections, and Quasiuniformities

Tom Richmond

After introducing some basic concepts and questions in the theory of ordered compactifications, the lattice of ordered compactifications of $X = [0,1] \times (\omega_1 \cup \{\omega_1 + 1\})$ is described. Each ordered compactification of X is determined by a pair (f, g) of functions on $[0,1] \cup \{\pm\infty\}$ which forms a Galois connection. The collection of all such functions f form the lower edges of entourages of a base for a quasiuniformity on $[0,1] \cup \{\pm\infty\}$. The topology and order on $[0,1]$ is recovered from this quasiuniformity after identifying $-\infty$ with 0 and 1 with ∞.

I would like to thank the Dagstuhl Foundation and the organizers of this Dagstuhl Seminar. This paper was inspired in part by conversations with Ralph Kummetz at Dagstuhl in June 2002. The diverse perspectives on topology and order in computational settings at this Dagstuhl Seminar promise to be very beneficial. Continuing collaboration here with Hans-Peter Künzi should soon result in a manuscript.

27 Mathematizing Existential Statements

Giovanni Sambin

A constructive treatment of subsets brings one naturally to the introduction of the notion of meet of two subsets, which is dual to inclusion: while $U \subseteq V$ is defined by a universal quantification $\forall x(x \in U \to x \in V)$, the new notion $U \between V$ is defined by an existetnial quantification $\exists x(x \in U \,\&\, x \in V)$.

Exploiting both $\subseteq$ and $\between$ one can give a simple description of existential and universal images of subsets along a relation. A surprising discovery is that the combinations of such images produce closure and interior operators, which coincide with the usual definitions of topological interior and closure. This is the beginning of what I call the basic picture, and the reason to introduce, in the definition of formal topology, a binary positivity predicate $\ltimes$ which is dual to the formal cover $\lhd$, just like $\between$ is dual to $\subseteq$.

In addition, the use of meet allows one to find an abstract characterization of relations in terms of functors. These, and other results, seem to suggest that the notions of $\between$ and $\ltimes$ can be the tools for a mathematical treatment of statements involving the existential quantifier.

28 Triangulations and Oriented Matroids

Francisco Santos

I decided to dedicate most of my talk to an introduction to oriented matroids, because most of the audience was unfamiliar with them and because there were several other talks (Rambau, Webster and, partially, Anderson) related to triangulations of oriented matroids. Apart of showing examples and giving one of the several (equivalent) axiomatic definitions of them (the definition by covectors) my main point was to make an analogy between the concept of oriented matroid and the concept of a topological space, in the following respects:

- Both topological spaces and oriented matroids appear in many different contexts, sometimes not explicitly.
- The definition of a topological space is axiomatic, and tries to capture the ideas of "neighborhood", "continuity" and "convergence" in metric spaces. The definition of oriented matroid is axiomatic, and tries to capture the ideas of "linear dependence" and "orientation" in a vector space over the reals (or over any other ordered field).
- In both cases, there are several different but equivalent axiom systems: open sets, closed sets, neighborhoods, closure, bases, etc, for topological spaces; circuits, cocircuits, covectors, vectors, chirotope, hyperline sequences, etc, for oriented matroids.
- In both cases, there are seemingly unwanted objects in the theory: non-metrizable spaces in topology, non-realizable oriented matroids (the ones which are not "the oriented matroid of a vector set in a real vector space") in oriented matroid theory. Whether these are pathological objects that one shouldn't care about or, on the contrary, make the theory more interesting, is a matter of opinion, or depends on the context.
- Unfortunately, both topological spaces and oriented matroids have a rather unappealing definition (some of us recall as a landmark in our mathematical life our first contact with the definition of a topological space, specially if, as happened in my University when I was undergraduate, this was given in the very first semester of your studies!!). But both become more and more natural as you start playing with them.
- As a preparation for Laura Anderson's results on the topology of triangulations of Euclidean oriented matroids, I also mentioned that both in topological spaces and in oriented matroids there is a whole hierarchy of more and more restrictive categories starting with just all the oriented matroids (resp. all the topological spaces) and ending with the realizable (resp. metrizable) ones. I am referring to the "separation axioms T_0, T_1, T_2, T_3, ... " in topology and the "intersection axioms IP_3, IP_2, IP_1 and IP_0" in oriented matroid theory. An oriented matroid is called Euclidean if it satisfies IP_3, the weakest of these axioms, which can be paraphrased as "every line and hyperplane intersect".

I devoted my final minutes to triangulations of oriented matroids. I recalled from Joerg Rambau's talk that the usual definition of triangulation for a finite point set in Euclidean space (or of simplicial fan of a finite vector set in a finite-dimensional real vector space) can be given in pure oriented matroid terms. This implies that the definition generalizes naturally to (perhaps non-realizable) oriented matroids, except for one thing: different oriented matroidal characterizations of triangulations in the realizable case may in principle not generalize to equivalent definitions in the non-realizable case. This problem is considered solved by the work of the author (Memoirs of the American Mathematical Society, number 741, 2002) in which the equivalence of seven such definitions is shown (but still, there is one definition not yet shown to be equivalent to those seven: the "circuit-admissible" triangulations of Rambau).

The main open problem now is the topological type of a triangulation of an oriented matroid. Every triangulation of a realizable oriented matroid of rank $r+1$ is topologically an r-ball or an r-sphere, depending only on whether the oriented matroid is "totally cyclic" or not. It is an open conjecture that the same happens in the non-realizable case. This conjecture would imply that the "combinatorial differential manifolds" of Gel'fand and MacPherson are truly topological manifolds. What we know so far is:

- Laura Anderson (1996) proved the conjecture for the class of Euclidean oriented matroids.
- The conjecture is true if restricted to the so-called lifting triangulations.
- Julian Webster (2002, preprint) has proved that triangulations have at least the same Euler characteristic, Dehn-Sommerville relations and upper bound theorem as polytopal spheres/balls.
- the author (2002, as yet unwritten) has proved that triangulations are simply-connected. By algebraic topological arguments this implies that they have the right homotopy type in rank up to 5 and, modulo Poincare's conjecture, the right topological type up to the same rank.

29 Coupled Lattices and Biframes

Anneliese Schauerte

Biframes are simultaneously generalizations of frames (locales) and bitopological spaces. Any quasi-uniform space has two natural underlying topologies, so bispaces arise naturally in this setting; biframes are localic versions of these. We establish the category equivalence of coherent biframes and coupled lattices and from it deduce the already known equivalence of Stone biframes and Boolean coupled lattices. Spatiality of these coherent biframes is equivalent to the Boolean Ultrafilter Theorem (a choice principle weaker than the Axiom of Choice). Their retracts are precisely the stably continuous biframes. A parallel theory can be obtained for supercoherent biframes, whose retracts (the stably supercontinuous biframes) are the semi-projective objects in the category of biframes.

30 Quantifying Domains via Semivaluations

Michel Schellekens

A characterization of partial metrizability is given which provides a partial solution to an open problem stated by Künzi in the survey paper "Nonsymmetric Topology". (Problem 7 "Characterize those quasi-uniformities having a countable base which are induced by a weighted quasi-metric".). The characterization yields a powerful tool which establishes a correspondence between partial metrics and generalized types of valuations, referred to as semivaluations.

As an application, we show that omega-continuous dcpo's are quantifiable, i.e. the Scott topology and partial order are induced by a partial metric. For omega-algebraic dcpo's the Lawson topology is induced by the associated metric. The partial metrization of general domains improves prior approaches in two ways:

- The partial metric is guaranteed to capture the Scott topology
- Partial metric spaces are Smyth-completable and hence their Smyth-completion reduces to the standard bicompletion.

Our proof of the quantifiability of domains is novel in that it relies on the central notion of a semivaluation. This notion was introduced in "The correspondence beween partial metrics and semivaluations" (TCS, to appear), where it was shown to underly various well known computer science applications. The present result are reported in "A characterization of partial metrizability. Domains are quantifiable" (TCS, to appear).

We conclude that the notion of a (semi)valuation is central in the context of Quantitative Domain Theory since it can be shown to underlie the various models arising in the applications.

31 Topological Relationships of Complex Points and Complex Regions

Markus Schneider

Topological relationships between spatial objects have been a focus of research on spatial data handling and reasoning for a long time. Especially as predicates they support the design of suitable query languages for spatial data retrieval and analysis in databases. Unfortunately, they are so far only applicable to simplified abstractions of spatial objects like single points, continuous lines, and simple regions, as they occur in systems like current geographical information systems and spatial database systems. Since these abstractions are usually not sufficient to cope with the complexity of geographic reality, their generalization is needed which especially has influence on the nature, definition, and number of their topological relationships. In the talk at the Dagstuhl Seminar we partially close this gap and first introduce very general spatial data types for complex points and complex regions. We then define the corresponding complete sets of mutually exclusive, topological relationships.

32 An Effective Version of Kisyński's Theorem

Matthias Schröder

Kisyński's Theorem states that every limit space $(X, \to_X)$ in which every converging sequence has a unique limit is *topological*, i.e. there is some topology on the set X inducing the convergence relation $\to_X$. We give an effective version of this theorem in the computational model of Type–2 Theory of Effectivity (TTE). Basic notions of TTE are *multirepresentation*, *relative computability*, and *continuous realizability*. A *multirepresentation* δ of a set X is a surjective correspondence between the Baire space $\mathbb{N}^{\mathbb{N}}$ and X. A function $f : X \to Y$ is called *computable* (*continuously realizable*) w.r.t. multirepresentations $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$ and $\gamma :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows Y$, iff there is a computable (continuous) function g on the Baire space such that $(\forall p \in \mathbb{N}^{\mathbb{N}})(\forall x \in \delta[p])\, f(x) \in \gamma[g(p)]$.
A multirepresentation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$ induces a final topology τ_δ and a final convergence relation $\rightsquigarrow_\delta$ on the set X; the latter is defined to be the finest convergence relation $\to$ on X such that $(X, \to)$ is a limit space and δ is sequentially contin-

uous. Every (δ,γ)–computable function $f : X \to Y$ turns out to be sequentially continuous w.r.t. the final convergence relations $\rightsquigarrow_\delta$ and $\rightsquigarrow_\gamma$ as well as w.r.t. the convergence relations induced by the final topologies τ_δ and τ_γ. The property of *admissibility* of multirepresentations is defined in such a way that every function f is continuously realizable, if and only if f is sequentially continuous w.r.t. the final convergence relations induced by the used multirepresentations. We define two operators $\cdot^{\mathcal{L}}$ and $\cdot^{\mathcal{T}}$ that map a multirepresentation δ to admissible ones of the generated quotient spaces $(X,\rightsquigarrow_\delta)$ and (X,τ_δ), respectively.
The effective version of Kisyński's Theorem states that the admissible multirepresentations $\delta^{\mathcal{L}}$ and $\delta^{\mathcal{T}}$ are computably equivalent (i.e. the identical function id_X is $(\delta^{\mathcal{T}},\delta^{\mathcal{L}})$–computable and $(\delta^{\mathcal{L}},\delta^{\mathcal{T}})$–computable), whenever the inequality test $\neq_X: X \times X \to \{\mathrm{ff},\mathrm{tt}\}$ is computable w.r.t. the product multirepresentation $\delta \otimes \delta$ and an effectively admissible multirepresentation of the Sierpiński space which has $\{\emptyset,\{\mathrm{tt}\},\{\mathrm{ff},\mathrm{tt}\}\}$ as its topology. As computability of the inequality test $\neq_X$ implies that every converging sequence has a unique limit in the generated limit space $(X,\rightsquigarrow_\delta)$ and since computable equivalence of $\delta^{\mathcal{L}}$ and $\delta^{\mathcal{T}}$ implies that τ_δ induces $\rightsquigarrow_\delta$, this effectivity result is actually an effective version of Kisyński's Theorem.

33 Ring Spectra Without Points

Peter M. Schuster

There may be not enough as well as too many prime ideals of a commutative ring whenever they are considered as points of the Zariski spectrum from a constructive and predicative standpoint, respectively. We try to solve this dilemma by avoiding points, or prime ideals, a strategy for which formal topology is most suited. Taking up the covering relation noticed by Persson, we attempt at completing the picture of a formal Zariski spectrum by proposing a binary positivity predicate, and a structure sheaf intended as a model for a notion of sheaves on formal topologies in general. In contrast to the localic version of the Zariski spectrum as presented by Johnstone, we formulate the universal property of it, as a geometric space, without mentioning points and stalks.

34 A Convenient Category of Domains

Alex Simpson

A convenient category of domains should have, at least, the following properties. It should support all the usual type constructors. It should allow recursive definitions both of data and types. It should support richer type features such as polymorphism. It should have the full range of powerdomains and, more generally, free algebras for (in)equational theories. Finally, it should have both continuous and effective versions.

In the talk, I present such a category which is also a full subcategory of topological spaces. The objects are monotone convergence spaces that are also quotients of countably-based spaces. The proof of convenience uses a detour through realizability models. It would be interesting to have direct topological accounts of the constructions.

35 Matroids from Modules

Micheal B. Smyth

The aim of this work is to show that matroid methods can be applied directly to typical discrete geometries (grid model, $\mathbb{Z}^n$), without the need to embed them in Euclidean space. The first observation is that a module over an integral domain has a matroid structure "in its own right". The flats of the matroid are not, however, the arbitrary submodules, but the d-submodules: those submodules of the given module M which are closed under those divisors which exist in M. (This construction seems to be known, though not well-known.) Next, we define oriented matroids in a way that allows these structures to be infinite, and show that every module over an ordered integral domain is an oriented matroid. An immediate corollary is that every abelian group is an oriented matroid.

We end by briefly considering topology in matroids. Every oriented matroid, in particular, has an intrinsic topology.

36 Admissible Representations of Convergence Spaces

Dieter Spreen

Weak limit spaces and convergences spaces are compared. Up to isomorphism the category of weak limit spaces is a reflective subcategory of the category of canonical convergences spaces. Both categories are Cartesian closed supercategories of the category of topological spaces and continuous maps.

Every open set in the induced topology of a canonical convergence space is sequentially open with respect to the induced notion of sequence convergence. The converse is true if the convergence space has a countable basis. In this case one can also introduce an admissible representation of the space (provided it is T_0 in the induced topology) such that the induced topology is the final topology of the representation and such that filter continuity of maps corresponds to continuous realizability.

37 Connection-Based Approaches to Space

John Stell

One way to formalize space is to take as primitives a set of 'regions' and a binary relation of 'connection' between regions. Intuitively two regions are connected if they touch or overlap. The development of axiomatizations of connection has a tradition going back at least to work of Whitehead and of de Laguna in the 1920s. Interest in the topic since the 1980s has been stimulated by work in artificial intelligence on qualitative spatial reasoning. In this talk I will start with an historical survey of the subject, and then describe recent developments in which various forms of 'connection algebra' have been investigated. The original motivation for the connection based approach was the description of continuous space, and I will discuss the formalization of discrete and finite spaces in terms of connection.

38 Compact Metric Spaces as Minimal Subspaces of Domains of Bottomed Sequences

Hideki Tsuiki

It is shown that when D is an ω-algebraic cpo for which the set $K(D)$ of finite elements of D is a finite-branching ω-type poset, the set $L(D)$ of limit elements of D has the set of minimal elements, which we denote by $M(D)$. When a Hausdorff space X is homeomorphic to $M(D)$ for some D, with the subspace topology of D on $M(D)$, and $M(D)$ is dense in D, every infinite ideal of $K(D)$ represents a unique point of $M(D)$ as the limit of a corresponding filter-base, and thus each infinite increasing sequence in the set $K(D)$ of finite elements of D expresses a point of X. Therefore, we have a representation of X. It is shown that it is a proper representation.

In the latter half of this talk, we show that when X is a compact metric space, there is a finite branching domain D such that X is homeomorphic to $M(D)$ and $M(D)$ is dense in D. Such a domain D can be chosen, when X is n-dimensional ($n = 0, 1, \ldots, \infty$), so that it consists of finite/infinite sequences of $\{0, 1\}$ in which at most n cells are allowed to be undefined ($\perp$), and therefore, through this embedding, one can view any compact metric space as a kind of space of infinite sequences with holes. Furthermore, minimality ensures that every infinite process which fills a tape following the finite elements of D represents a unique point of X, and thus computation over the space X is defined through a variant of an IM2-machine ([1]).

References:

[1] Hideki Tsuiki. Real number computation through gray code embedding. *Theoretical Computer Science*, 2002. to appear.

[2] Hideki Tsuiki. Computational dimension of topological spaces. In Jens Blanck, Vasco Brattka, and Peter Hertling, editors, *Computability and Complexity in Analysis*, volume 2064 of *Lecture Notes in Computer Science*, pages 323–335, Berlin, 2001. Springer.

39 Uncertain Geometric Relations in Digital Images

Peter Veelaert

Uncertain geometry has been developed to determine what geometric relations exist between line segments in a digital image. To cope with the uncertainity introduced by digitization and line detection algorithms, in uncertain geometry we associate with each point or pixel an uncertainty region. Then a set of points is called straight if there exists a Euclidean straight line crossing all the uncertainty regions of the points in the set. Parallelism, collinearity and concurrency are defined in a similar way. For example, two point sets are called parallel if we can find two parallel Euclidean straight lines, such that for each set, one of the Euclidean lines crosses all the uncertainty regions associated with that set. Thus uncertain geometry inherits all its definitions in a straightforward way from Euclidean geometry. An apparent drawback of this approach is that it leads to inconsistencies, due to the introduction of uncertainty. In previous work we have illustrated how these geometric consistency can be restored. First, at the combinatorial level we can restore consistency by using grouping algorithms which are based on Helly's Theorem. This corresponds to consistency at the level of the incidence relations of Euclidean Geometry. Second, properties at a somewhat higher level such as Desargues's or Pappus's Theorem can be restored during the grouping process by gradual reductions of the uncertainty of the line parameters. It is still an open question how we can integrate the results obtained in uncertain geometry with concepts that deal with topological uncertainty, such as Region Connection Calculus, rough sets and fuzzy sets.

40 The Double Powerlocale

Steve Vickers

If X is a locale then its double powerlocale $PP(X)$ is defined as $P_U(P_L(X))$, where P_U and P_L are the upper (Smyth) and lower (Hoare) powerlocale constructions. By a result of Johnstone and Vickers, it is homeomorphic to $P_L(P_U(X))$ and its frame $\Omega(PP(X))$ is generated by $\Omega(X)$ preserving its dcpo (directed complete partial order) structure.

I describe various properties of the double powerlocale.

1. PP is the functor part of a monad on the category of locales.

2. If X is locally compact, hence exponentiable, then PP(X) is homeomorphic to the exponential locale $\mathbb{S}^(\mathbb{S}^X)$ where $\mathbb{S}$ is the Sierpinski locale.

3. Current work with Christopher Townsend aims to show that the isomorphism between $PP(X)$ and $\mathbb{S}^{(}\mathbb{S}^{X})$ can be shown even if X is not locally compact, using the Yoneda embedding of the category of locales in a presheaf category.

4. Using this, the strength $\chi : PP(X) \times Y \to PP(X \times Y)$ would be defined as

$$\chi(U, y) = \lambda a.(U(\lambda x.a(x, y)))$$

This leads to an independent definition of its inverse image function, satisfying

$$\chi^*([] <> \bigvee_{i=1}^{n}(a_i \times b_i)) = \bigvee\{[] <> \bigvee_{i\in S} a_i \times \bigwedge_{i\in S} b_i \mid S \text{ subset of } \{1, \dots, n\}\}$$

($[] <>$ constructs generators of $\Omega(PP(X))$ out of elements of $\Omega(X)$.)

5. Till Plewe defines a map of locales $f : X \to Y$ to be triquotient if it is equipped with a Scott continuous function $f_\sharp : \Omega(X) \to \Omega(Y)$ satisfying two Frobenius conditions. Such a map is always a localic surjection. Triquotient maps include all open surjection and all proper surjections, and share many properties with them. I define generalized triquotient maps, containing all open maps and all proper maps, equipped with Scott continuous f_sharp satisfying two generalized Frobenius conditions

$$f_\sharp(a \wedge f^*(b)) = (f_\sharp(a) \wedge b) \vee f_\sharp(0)$$
$$f_\sharp(a \vee f^*(b)) = (f_\sharp(a) \vee b) \wedge f_\sharp(1)$$

A map $f : X \to Y$ can be shown to be generalized triquotient iff it is equipped with a map $f^o : Y \to PP(X)$ such that the two composites

$$f^o; PP(< X, f >) : Y \to PP(X) \to PP(X \times Y)$$
$$< f^o, Y >; \chi : Y \to PP(X) \times Y \to PP(X \times Y)$$

are equal.

41 An Overview of the Constructive Theory of Apartness Spaces

Luminita Simona Vita[5]

[5]joint work with Douglas Bridges

Apartness spaces are regarded as a suitable constructive framework for general topology. A set-set apartness relation between subsets of a given nonempty set is introduced via an axiomatic system. The corresponding point-set apartness is derived form the set-set one and is used to define the apartness topology. Uniform spaces are discussed as a basic model of apartness space.

The classical theory of proximity spaces (of which apartness spaces are the constructive analogue) shows that every such space is given by a compatible uniformity. Constructively there exist point-set apartnesses that can be extended to set-set apartnesses which are not uniformizable; hence the theory of apartness spaces is strictly bigger than that of uniform spaces. Various notions of continuity for mappings between apartness spaces and the relationships between them are investigated.

42 Riemann Type Intergrals on Compact Hausdorff Measure Spaces

Guojun Wang

Let $(X, \mathcal{T})$ be a compact Hausdorff-space, then there is a uniformity $\mathcal{U}$ on $X \times X$ such that $\mathcal{U}$ induces $\mathcal{T}$. Suppose that $(X, \mathcal{A}, \mu)$ is a measure space such that the family $\mathcal{B}(X)$ of all Borel subsets of $(X, \mathcal{T})$ is a subset of $\mathcal{A}$, and$\forall E \in \mathcal{A}$, if $\mu(E) = 0$, then $\forall \epsilon > 0$ there exists an open subset G s.t. $E \subset G$ and $\mu(G) < \epsilon$. Let $\mathcal{U}^0$ be the subfamily of $\mathcal{U}$ s.t. $\mathcal{U}^0 = \{U \in \mathcal{U} \mid U$ is symmetric and open in $(X, \mathcal{T})^2\}$.

A partition $\mathcal{P} = \{P_1, \cdots, P_n\}$ of X is called a U-Partition ($U \in \mathcal{U}^0$) if there exists an oper cover $\mathcal{C} = \{C_1, \ldots, C_n\}$ of X s.t. $P_1 = C_1, P_k = C_k - \bigcup_{j=1}^{k-1} C_j$ $(k = 2, \ldots, n)$, and $C_i \times C_i \subset U$ $(i = 1, \cdots, n)$.

Assume that $\mu(X) = M < +\infty$. Let f be any bounded measurable real function. Define

1. $S^-(\mathcal{P}, f) = \sum_{P_i \in \mathcal{P}} \mu(P_i) \inf f(P_i)$, $S^+(\mathcal{P}, f) = \sum_{P_i \in \mathcal{P}} \mu(P_i) \sup f(P_i)$.
2. $S^-(U, f) = \inf\{S^-(\mathcal{P}, f) \mid \mathcal{P}$ is a U-partition$\}$,
   $S^+(U, f) = \sup\{S^+(\mathcal{P}, f) \mid \mathcal{P}$ is a U-partition$\}$.
3. $(R) \int^- f \, d\mu = \sup\{S^-(U, f) \mid U \in \mathcal{U}^0\}$, $(R) \int^+ f d\mu = \inf\{S^+(U, f) \mid U \in \mathcal{U}^0\}$.

f is R-integrable iff $(R)\int^{-} f d\mu = (R)\int^{+} f d\mu$. This paper proves that

1. f is R-integrable iff f is continuous almost everywhere.
2. If f is R-integrable then f is L-integrable and $(R)\int f d\mu = (L)\int f d\mu$.
3. If $X = \mathbb{R}^n$, then $(R)\int f d\mu$ coincides with Riemann integral of f.
4. If $(X, \mathcal{T})$ is a compact metric space, then there exists a constructive structure for calculating R-integrals.

43 Partial Semimetric Spaces in Domain Theory and Topology

Pawel Waszkiewicz

We introduce a general notion of distance in weakly separated topological spaces. Our approach differs from the existing ones since we do not in general assume the reflexivity axiom. We demonstrate that our partial semimetric spaces provide a common generalization of semimetrics known from Topology and both partial metrics and measurements studied in Quantitative Domain Theory. We focus on the local axiom of triangle, which is a substitute for the triangle inequality in our distance spaces. We use this property to prove a counterpart of the famous Archangelskij metrization theorem in the more general context of partial semimetric spaces. Finally, we consider the framework of algebraic domains and employ Lebesgue measurements to obtain a complete characterization of partial metrizability of the Scott topology.

44 Oriented Matroids and Finite Spatial Representation

Julian Webster

An oriented matroid is an axiomatic combinatorial structure that captures the affine and convex structure of a finite point set in Euclidean space. With regard to finite representation of a *region*, such as the unit square, we consider triangulated oriented matroids, which are in a precise sense purely combinatorial versions of triangulated Euclidean polytopes. We propose that triangulated oriented matroids be considered as an axiomatic basis for digital geometry. In investigating this geometry we have found that:

1. Every triangulation of every oriented matroid is partitionable;
2. Every triangulation of every totally cyclic oriented matroid satisfies the Dehn-Sommerville equations;
3. For any simplex S in any triangulation of any oriented matroid $\mathcal{M}$ such that S is not contained in a facet of $\mathcal{M}$, $link(S)$ satisfies the Dehn-Sommerville equations.

As is well-known, any simplicial complex satisfying the Dehn-Sommerville equations also satisfies the Euler-Poincaré relation and the Upper Bound theorem.

45 Digital Spaces Constructed from R^n by Non-Convex Tiles

Petra Wiederhold de Matos and Richard Wilson

A digital space (due to Kronheimer, 1992) is a special open quotient space X of R^n constructed from a family $\mathcal{W}$ (called fenestration) of pairwise disjoint regular open sets in R^n whose union is dense in R^n. If $\mathcal{W}$ is locally finite, then X can be considered as constructed from the corresponding tiling (the family of closures of the elements of $\mathcal{W}$) as well. For locally finite fenestrations, the digital space is an Alexandroff T_0 space. We deal with the problem, for which locally finite fenestrations of R^n, the digital space X has dimension n.

We had proved that for any Alexandroff T_0 space, its small inductive dimension coincides with the partial order dimension $Odim$ of $(X, \leq)$, where $\leq$ is the specialization order of X, and $Odim(X) = sup\{| C |: C$ is a chain of distinct elements in $(X, \leq)\}$ (in: *Vision Geometry*, ed. by Melter/Wu, SPIE Proc. 1832, 13-22, 1993). For general locally finite fenestrations of R^n, the digital space fails to have dimension n. We proved that if for a locally finite fenestration $\mathcal{W}$ of R^n, each element of $\mathcal{W}$ is bounded and convex, then X has dimension n, (*Discrete and Comput. Geometry*, 27, 273-286, 2002). Now we prove, by applying a canonical construction and purely topological arguments, that if $\mathcal{W}$ is a locally finite fenestration of R^2, where for each $W \in \mathcal{W}$, W is homeomorphic to an open disc B, and $fr(W) \cong fr(B)$, then X has dimension 2. The construction cannot be applied for similar fenestrations of R^n for $n \geq 3$.

It remains an open problem how to contruct the digital space in the non-convex case for higher dimensions.

46 Stable Bifinite Domains: The Finite Antichain Condition

Guo-Qiang Zhang[6]

Amadio and Curien raised the question of whether the category of stable bifinite domains is the largest cartesian closed full sub-category of the category of ω-algebraic meet-cpos with stable functions. We provide a progress report on our understanding of the situation, in part using a characterization of traces developed for this purpose. This characterization states that a set of compact-element pairs is the trace of a stable function if and only if it is joinable. The joinability condition offers intuition as well as another possible way for the creation of stable functions, a step lying at the heart of maximality proofs. We show that for an ω-algebraic meet-cpo D, if the higher-order stable function space $[D \to D] \to [D \to D]$ is 2/3 SFP, then D must be finitary. This is a crucial step forward in settling Amadio and Curien's question.

[6] joint work with Jiang Ying and Yixiang Chen