Similarity quotients as final coalgebras

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Outline

Examples

We study the following examples:

1. bisimilarity
2. bisimilarity and similarity together
3. similarity
4. upper similarity
5. intersection of lower and upper similarity
6. 2-nested similarity

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2. bisimilarity and similarity together
3. similarity
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6. 2-nested similarity

In each case we see

- how to use a final coalgebra
- how to construct a final coalgebra.

Bisimilarity: Using A Final Coalgebra

Fix a countable set Act of labels.

Let $F : X \mapsto \mathcal{P}^{\leqslant\aleph_0}(\mathrm{Act} \times X)$ on **Set**.

A countably branching Act-labelled transition system is an F-coalgebra.

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Theorem: no junk

Every element of A is of the form $\sigma_B b$.

Bisimilarity: Constructing A Final Coalgebra

Let $F : X \mapsto \mathcal{P}^{\leqslant\aleph_0}(\mathsf{Act} \times X)$ on **Set**.

Suppose A is a transition system that is big enough
i.e. every $b \in B$ is bisimilar to some $a \in A$.

Then A modulo bisimilarity (with behaviour map chosen to make $A \longrightarrow A/\eqsim$ a homomorphism) is a final F-coalgebra.

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Example of a big enough transition system

$A \stackrel{\mathrm{def}}{=}$ the disjoint union of all transition systems on initial segments of $\mathbb{N}$.
It's big enough because every (B, b) has countably many successors.

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If A isn't big enough, then $A/\simeq$ is still subfinal, i.e. parallel morphisms to it are equal.

Bisimilarity and Similarity: Using A Final Coalgebra

Let G be the endofunctor on **Preord** mapping $(X, \leqslant)$ to

$$(\mathcal{P}^{\leqslant\aleph_0}(\mathsf{Act} \times X), \mathsf{Sim}(\leqslant))$$

where $U \ \mathsf{Sim}(\mathcal{R})\ V \stackrel{\text{def}}{\Leftrightarrow} \forall x \in U. \exists y \in V. u\ \mathcal{R}\ v$.

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Quotienting by a preorder

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So **Poset** is a full reflective subcategory of **Preord**.

$$\mathbf{Poset} \underset{\hookrightarrow}{\overset{Q}{\underset{\perp}{\longleftarrow}}} \mathbf{Preord}$$

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Then A modulo similarity is a final G-coalgebra.

Lower and Upper Simulations (Ulidowski, Lassen, Pitcher)

A countably branching Act-labelled transition system with divergence is a coalgebra for $X \mapsto \mathcal{P}^{\leqslant \aleph_0}((\mathsf{Act} \times X)_\perp)$.

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Let B and C be two such, and let $\mathcal{R} \subseteq B \times C$ be a relation.

Lower simulation

$\mathcal{R}$ is a lower simulation when, for $b \ \mathcal{R} \ c$

- $b \overset{a}{\leadsto} b'$ implies there is c' such that $c \overset{a}{\leadsto} c'$ and $b' \ \mathcal{R} \ c'$.

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There are many variants.

Upper similarity and final coalgebras

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$$U \ \mathsf{Upper}(\mathcal{R}) \ V \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \perp \notin U \Rightarrow (\perp \notin V \wedge \forall y \in V.\ \exists x \in U.\ x \ \mathcal{R} \ y)$$

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Then

- a final H-coalgebra characterizes upper similarity, with no junk
- a big enough transition system with divergence, modulo upper similarity, gives a final H-coalgebra.

Doubly preordered sets

We are going to study the intersection of lower similarity and upper similarity.

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We obtain categories

$$\mathbf{TwoFine} \underset{\longleftarrow}{\overset{Q}{\underset{\bot}{\longleftarrow}}} \hookrightarrow \mathbf{TwoPreord}$$

Lower similarity ∩ upper similarity

Let *G* be the endofunctor on **TwoPreord** mapping $(X, \leqslant_l, \leqslant_u)$ to

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2-nested simulation (Groote and Vaandrager)

Let B and C be transition systems.

A 2-nested simulation from B to C is a simulation contained in the converse of a simulation.
Equivalently a simulation contained in the converse of similarity.
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Corresponds to modal formulas $\Diamond^n$ and $\Diamond^n\Box^m$.

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$$\mathbf{NestFine} \underset{\hookrightarrow}{\overset{Q}{\underset{\perp}{\longleftarrow}}} \mathbf{NestPreord}$$

2-nested simulation and final coalgebras

Let G be the endofunctor on **NestPreord** mapping $(X, \leqslant_1, \leqslant_2)$ to

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Then

- a final H-coalgebra characterizes (the converse of) similarity and 2-nested similarity, with no junk
- a big enough transition system, modulo 2-nested similarity, gives a final H-coalgebra.

Proving these results simultaneously

Instead of having to prove all these results separately,

we want general theorems that they are all instances of.

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What is the data for our theorems?

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Examples with $\mathcal{C} = \mathbf{Set}$

A quasi-relation $A \xrightarrow{\mathcal{R}} B$ is

1. a relation
2. a pair of relations $(\mathcal{R}_l, \mathcal{R}_u)$
3. a pair of relations $(\mathcal{R}_1, \mathcal{R}_2)$ with $\mathcal{R}_2 \subseteq \mathcal{R}_1$.

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We then define "quasi-preordered objects" and "finely quasi-preordered objects".

$$\mathbf{Fine}(\mathcal{A}) \underset{\hookrightarrow}{\overset{Q}{\underset{\perp}{\longleftarrow}}} \mathbf{Preord}(\mathcal{A})$$

Relational extension

We require an endofunctor F on $\mathcal{C}$, expressing the behaviour

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$$\begin{array}{rcl} X & \mapsto & \mathcal{P}^{\leqslant \aleph_0}(\mathsf{Act} \times X) \\ X & \mapsto & \mathcal{P}^{\leqslant \aleph_0}((\mathsf{Act} \times X)_\perp) \end{array}$$

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And we need Γ mapping a quasi-relation $X \xrightarrow{\mathcal{R}} Y$ to $FX \xrightarrow{\Gamma\mathcal{R}} FY$.

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And we need Γ mapping a quasi-relation $X \xrightarrow{\mathcal{R}} Y$ to $FX \xrightarrow{\Gamma\mathcal{R}} FY$.

Γ is a relational extension of F.

Properties of relational extension (1)

Monotonicity

$$X \xrightarrow{\mathcal{R},\mathcal{R}'} Y$$

$$\mathcal{R} \subseteq \mathcal{R}' \Rightarrow \Gamma\mathcal{R} \subseteq \Gamma\mathcal{R}'$$

Properties of relational extension (1)

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Stability (Hughes and Jacobs)

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ & & \downarrow \mathcal{R} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

$$\Gamma((f \times g)^{-1}\mathcal{R}) = (Ff \times Fg)^{-1}\mathcal{R}$$

Properties of relational extension (2)

Lax functoriality

$$
\begin{aligned}
\mathrm{id}_{\Gamma X} &\subseteq \Gamma \mathrm{id}_X \\
(\Gamma\mathcal{R});(\Gamma\mathcal{S}) &\subseteq \Gamma(\mathcal{R};\mathcal{S})
\end{aligned}
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$$
X \xrightarrow{\mathcal{R}} Y \xrightarrow{\mathcal{S}} Z
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Hesselink and Thijs required strict preservation of binary composition, lax preservation of identities.

But that excludes the case of 2-nested simulation.

Conclusions

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Theorems

- any final H-coalgebra characterizes Γ-simulation, and has no junk
- any coalgebra that is big enough, modulo Γ-similarity, gives a final H-coalgebra.