# Stone duality and canonical extensions

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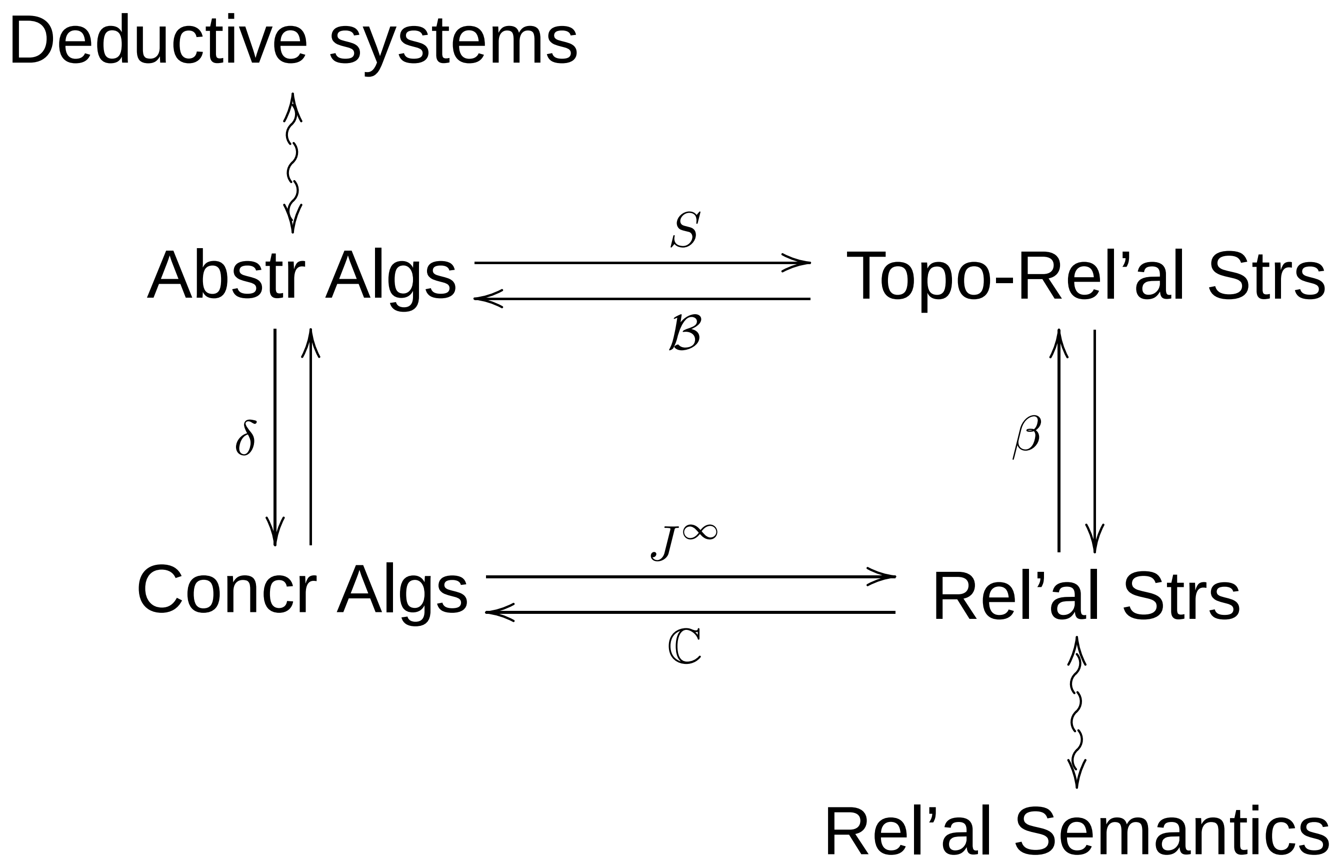
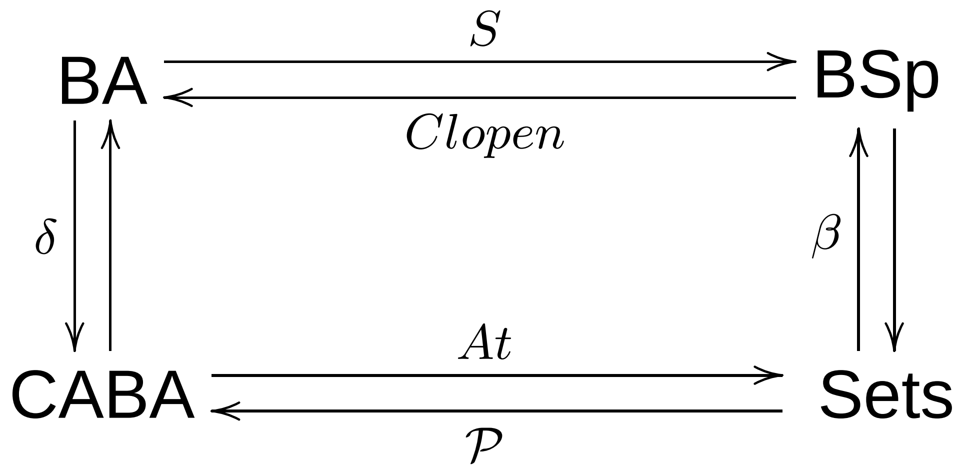
# Duality for logic



# Boolean duality

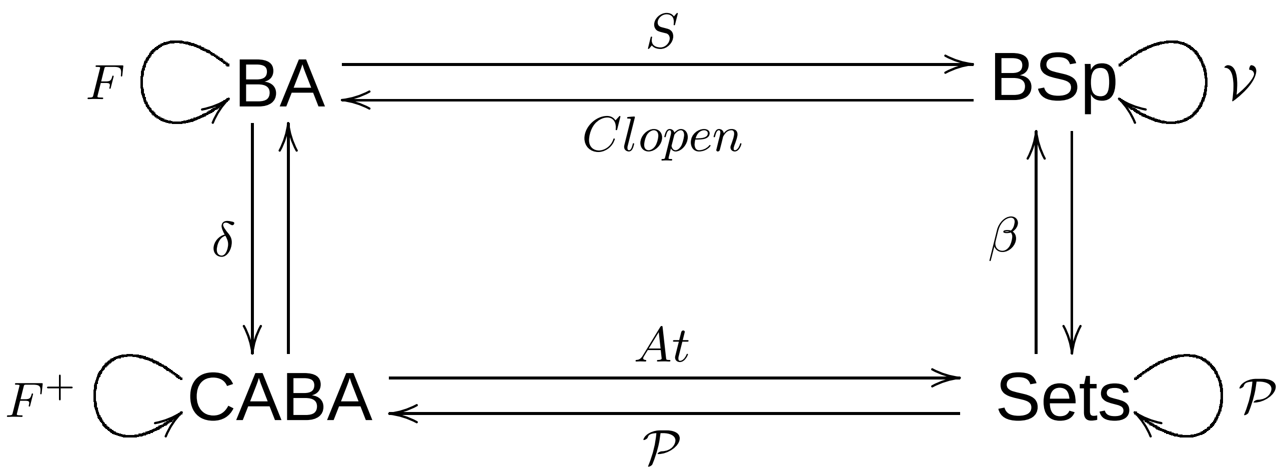



The unit of the adjunction contains the same information as the dual space

$$(X_A, < \widehat{a} \mid a \in A >) \quad\quad \leftrightsquigarrow \quad\quad \eta_A : A \hookrightarrow \mathcal{P}(X_A)$$

The latter is the canonical extension of Jónsson and Tarski (1951)

# Additional connectives




NB!!! The up/down functors are no longer adjunctions for the algebras/co-algebras [Goldblatt 2006]

Canonical extension is still the dual incarnation of the forgetful functor

# Canonical extension

is particularly well suited for treating:

- subvarieties
- order-reversal (more general additional operations)
- more general underlying posets

# Canonical extension

The reflector

$$\text{BA} \leftrightarrows \text{CABA}$$

given by

$$\eta_A : A \hookrightarrow \mathcal{P}(X_A) = A^\delta$$

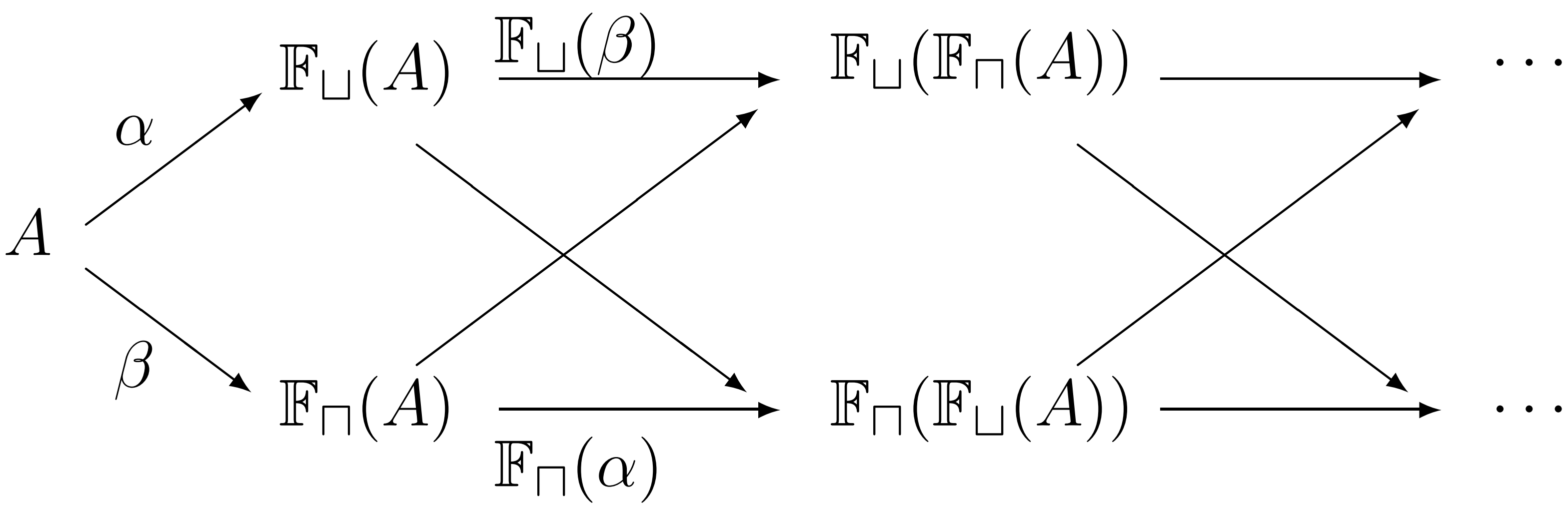
for each $A$ is characterized by

- $\eta_A$ is a completion of $A$
- $\eta_A$ is $\Delta_1$-dense: everything in $A^\delta$ is a join of meets and a meet of joins of elements from $A$
- $\eta_A$ is compact: For $S, T \subseteq A$ we have $\bigwedge S \leq \bigvee T$ implies $\bigwedge S' \leq \bigvee T'$ for some finite $S' \subseteq S$ and $T' \subseteq T$

# Completions hierarchy

[G-Priestley 2008]

For a lattice $A$ we have




$\Delta_0$-completion$(A)$ = Dedekind-Mac Neille completion$(A)$

$$\begin{aligned}\Delta_1\text{-completion}(A) &= \eta_A : A \hookrightarrow A^\delta \\ &= \text{canonical extension}(A)\end{aligned}$$

# Dcpo presentation

[Jung-Moshier-Vickers 2008]

A dcpo presentation is a preorder $\langle P, \sqsubseteq \rangle$ with a cover relation $\lhd \subseteq P \times \mathcal{P}_{Dir}(P)$ ...

[G-Vosmaer]

Given a lattice $A$, the canonical extension of $A$ is presented by

$$\Delta(A) = \langle\, \mathbb{F}_{\sqcap}(A)\, ;\, \leq\, ,\, \lhd_A \,\rangle$$

where $F \lhd_A U$ if and only if,

$$\forall i \in Idl(A) \;\; \left[\forall\, f' \in U \;\, i \cap f' \neq \emptyset \quad \Rightarrow \quad i \cap f \neq \emptyset\right]$$

for each non-empty directed $U \subseteq \mathbb{F}_{\sqcap}(A)$ and $f \in \mathbb{F}_{\sqcap}(A)$

# Extension of additional operations

[G-Jónsson]

Both $\mathbb{F}_\sqcap(A)$ and $\mathbb{F}_\sqcup(A)$ embed in $A^\delta$ and the topology $\delta$ generated by $[f, i]$ with $f \in \mathbb{F}_\sqcap(A)$ and $i \in \mathbb{F}_\sqcup(A)$ is Hausdorff and the isolated points are exactly the elements of $A$

Maps $g : A \to B$ extend by semicontinuous envelopes:

$$g^\sigma(u) = \underline{\lim} g(u) = \bigvee\{\bigwedge g([f, i] \cap A) \mid u \in [f, i] \in \delta\}$$

$$g^\pi(u) = \overline{\lim} g(u) = \bigwedge\{\bigvee g([f, i] \cap A) \mid u \in [f, i] \in \delta\}$$

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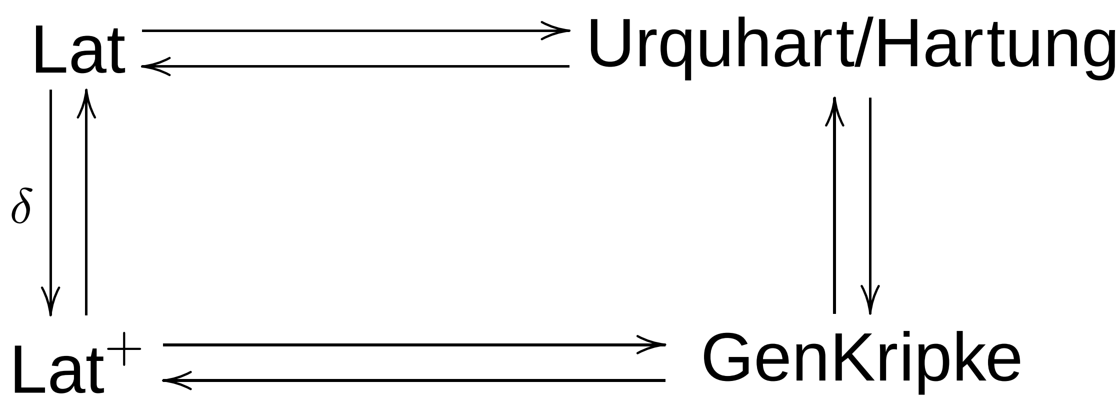
# Lattice duality




Generalized Kripke structures are $(X, Y, \leq)$ where

- $X =$ completely join irreducibles of $L^\delta$ (worlds)
- $Y =$ completely meet irreducibles of $L^\delta$ (irred. info bits)
- $\leq \subseteq X \times Y$ ($y$ is available at $x$)

# Generalized Kripke Structures

- Modular approach to completeness for basic substructural hierarchy [Dunn-G-Palmigiano 2005]
- Connections to proof-theory [Galatos-Jipsen]
- Connection to Yde & Co' s $\nabla$?

# References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[Jónsson-Tarski 1951] *Amer. J. Math.* **73**, 891–939

[Jung-Moshier-Vickers 2008] *Electr. Notes Theor. CS* **218**, 209–229