

Negating Congruences for Stochastic Relations

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NEGATION

Tricky problem

- 1 “Die Welt ist alles, was der Fall ist” (L. Wittgenstein, *Tractatus*) - things are so positive.
- 2 Negation in data bases: leads to the *closed world assumption*.

ADVERSE

Take an equivalence relation ρ on a {set, analytic space} X ; express the negation of ρ through another equivalence relation ρ' , its *adverse*. Thus:

$$\rho \wedge \rho' = \Delta_X, \rho \vee \rho' = \nabla_X.$$

REMARK

The equivalence relations of interest are induced by some modal or coalgebraic logic; they are **countably generated** (sometimes called **smooth** or **tame**).

PROPOSITION

Each countably generated (cg) equivalence ρ on an analytic space has a countably generated adverse ρ' .

LEMMA

The map $\rho \mapsto \Sigma(\rho)$ which maps each countably generated equivalence relation to the σ -algebra of its invariant Borel sets is an antitone isomorphism.

PROOF OF THE PROPOSITION

There exists for $\Sigma(\rho)$ a countably generated σ -algebra \mathcal{D} with $\Sigma(\rho) \vee \mathcal{D} = \mathcal{B}(X)$ and $\Sigma(\rho) \wedge \mathcal{D} = \{\emptyset, X\}$ (Rao&Rao). Then $\mathcal{D} = \Sigma(\rho')$.

STOCHASTIC RELATIONS

A stochastic relation $K : X \rightsquigarrow Y$ is a Kleisli morphism for the Giry monad (thus $K(x) \in \mathbb{S}(Y)$ is a subprobability on Y for each x , and $x \mapsto K(x)(A)$ is measurable for each A).

CONGRUENCES

A **congruence** (α, β) for $K : X \rightsquigarrow Y$ is a pair of cg equivalence relations on X and Y such that $x_1 \alpha x_2 \Rightarrow K(x_1)(A) = K(x_2)(A)$ for all β -invariant measurable sets $A \subseteq Y$.

(α, β) is a congruence for K iff $\alpha \subseteq \ker(K^{\uparrow\beta})$, where $K^{\uparrow\beta}(x)$ is the restriction of $K(x)$ to $\Sigma(\beta)$.

STRUCTURE?

What can we say about the order structure of the set of all congruences?

ATOMS

The **atoms** in the set of all cg equivalences on an analytic space X are of the form $\iota_{a,b} := \Delta_X \cup \{\langle a, b \rangle, \langle b, a \rangle\}$ for $a \neq b$.

ANTIATOMS

The **antiatoms** look like this:

$$x \beta_A x' \text{ iff } \{x, x'\} \subseteq A \text{ or } \{x, x'\} \cap A = \emptyset.$$

(A Borel with $\emptyset \neq A \neq X$)

EED.

REPRESENTATION THROUGH ANTIATOMS

Given a cg equivalence β , there exists a sequence $(\beta_n)_{n \in \mathbb{N}}$ of antiatoms with $\beta = \bigwedge_{n \in \mathbb{N}} \beta_n$.

CONSEQUENCE

If β is a cg equivalence on Y with $K : X \rightsquigarrow Y$, then there exists a sequence $(\beta_n)_{n \in \mathbb{N}}$ of **antiatoms** with

$$\ker(K \uparrow \beta) = \bigwedge_{F \subseteq_{\text{fin}} \mathbb{N}} \ker(K \uparrow \bigwedge_{f \in F} \beta_f)$$

REMARK

A similar representation through **atoms** holds iff the base space is at most countable.

EED.

CONSEQUENCE

If β is a cg equivalence on Y with $K : X \rightsquigarrow Y$, then there exists a sequence $(\beta_n)_{n \in \mathbb{N}}$ of antiatoms with

$$\ker(K^{\uparrow\beta}) = \bigwedge_{F \subseteq_{\text{fin}} \mathbb{N}} \ker(K^{\uparrow \bigwedge_{f \in F} \beta_f})$$

EXTREME POINTS

The extreme points in the set of all congruences for $K : X \rightsquigarrow Y$ are the pairs $(\iota_{x,x'}, \beta_A)$ with

- 1 $x \neq x'$,
- 2 $A \subseteq Y$ Borel with $\emptyset \neq A \neq Y$,
- 3 $K(x)(A) = K(x')(A)$ and $K(x)(Y) = K(x')(Y)$.

CONCLUSION

EED.

ADVERSE

We had a look at finding an equivalence relation which negates a given one.

$$\rho \leftrightarrow \Sigma(\rho)$$







The interplay of cg equivalence relations and their invariant sets was used for shedding some light on the structure of congruences for a stochastic relation.

DOMAINS?

We would like to know more about the interplay between order and structure for stochastic relations. Can we define sensible domains?

SOME POINTERS

EED.

-  S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer-Verlag (The Millennium Edition), 1981.
-  E.-E. Doberkat, *Stochastic coalgebraic logic*, EATCS Monographs in Theoretical Computer Science, Springer-Verlag, 2009.
-  E.-E. Doberkat and Ch. Schubert, *Coalgebraic logic for stochastic right coalgebras*, Ann. Pure Appl. Logic **159** (2009), 268–284.
-  A. S. Kechris, *New directions in descriptive set theory*, Bull. Symb. Logic **5** (1999), no. 2, 161 – 174.
-  K. P. S. Bhaskara Rao and B. V. Rao, *Borel spaces*, Dissertationes Mathematicae **190** (1981), 1–63.
-  S. M. Srivastava, *A course on Borel sets*, Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1998.