Tight Bounds for Blind Search on the Integers

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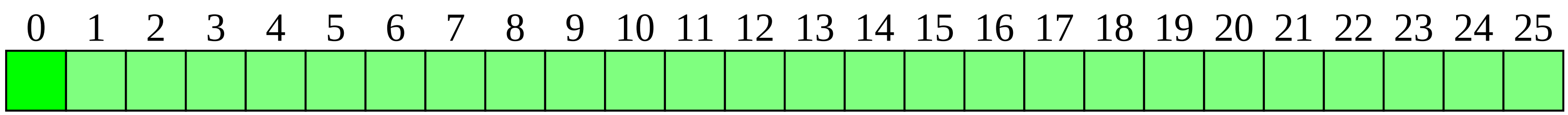
Dagstuhl, September 18, 2008

A simple board game



“Random walk” on “board” $\{0, 1, \ldots, n\} = [0, n]$

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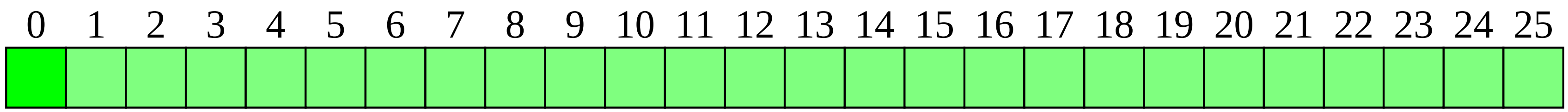

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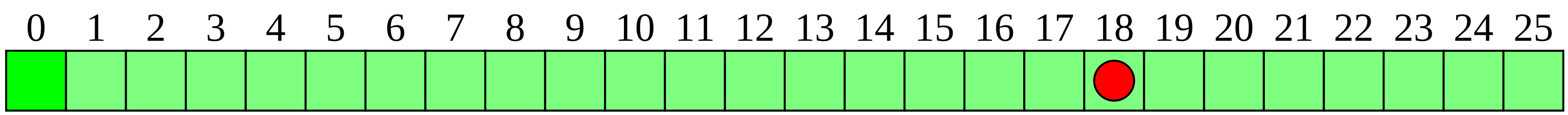

Start: **Uniformly random** position R_0 in $[1, n]$

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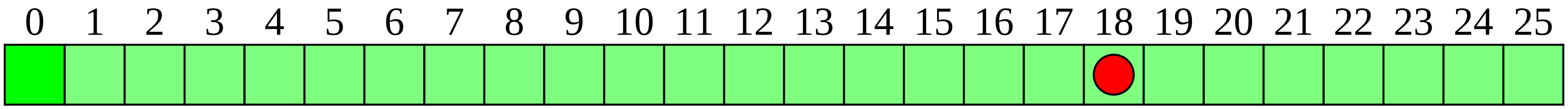

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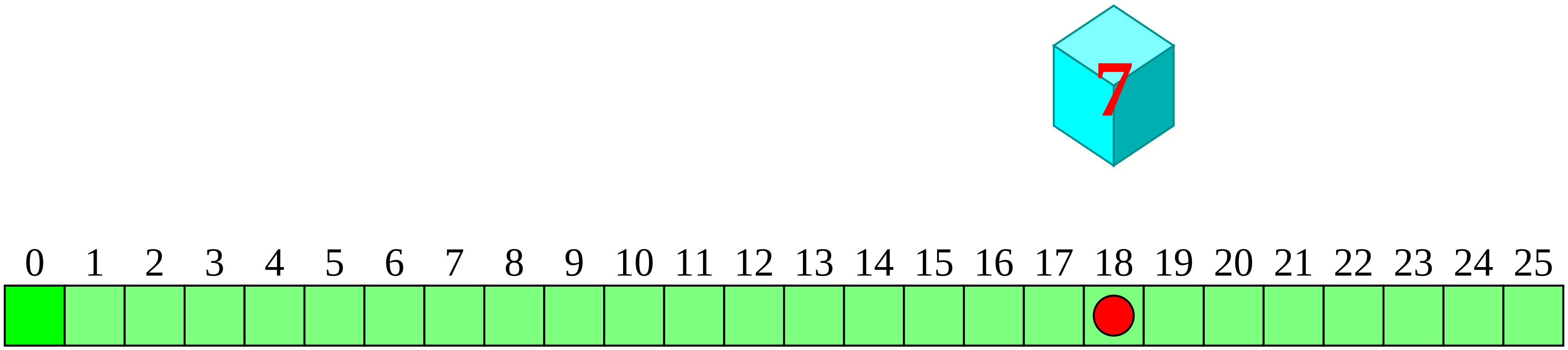

Step t: Given position R_{t-1}, choose "distance" D_t from $[1, n]$, according to **probability distribution** $\boldsymbol{\mu}$ on $[\mathbf{1}, \boldsymbol{n}]$.

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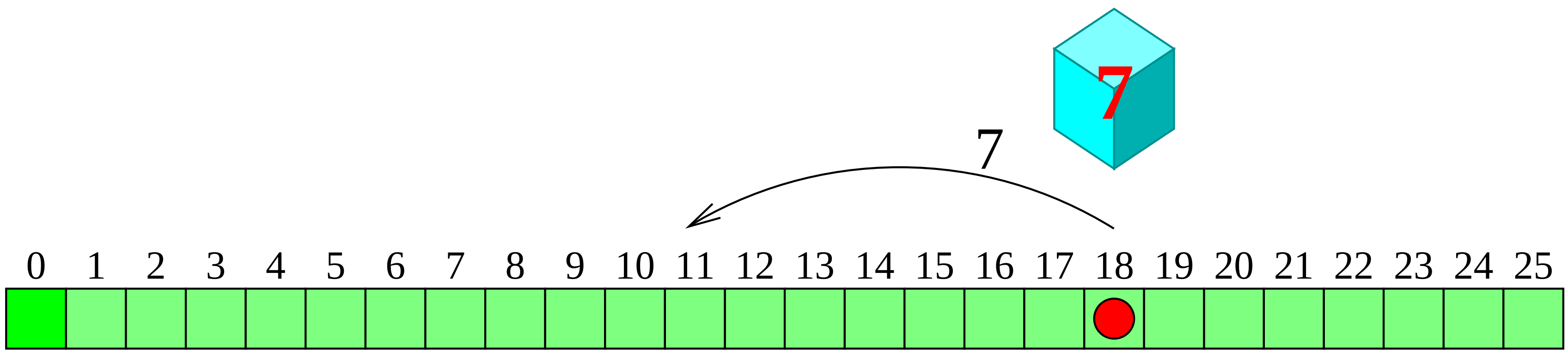

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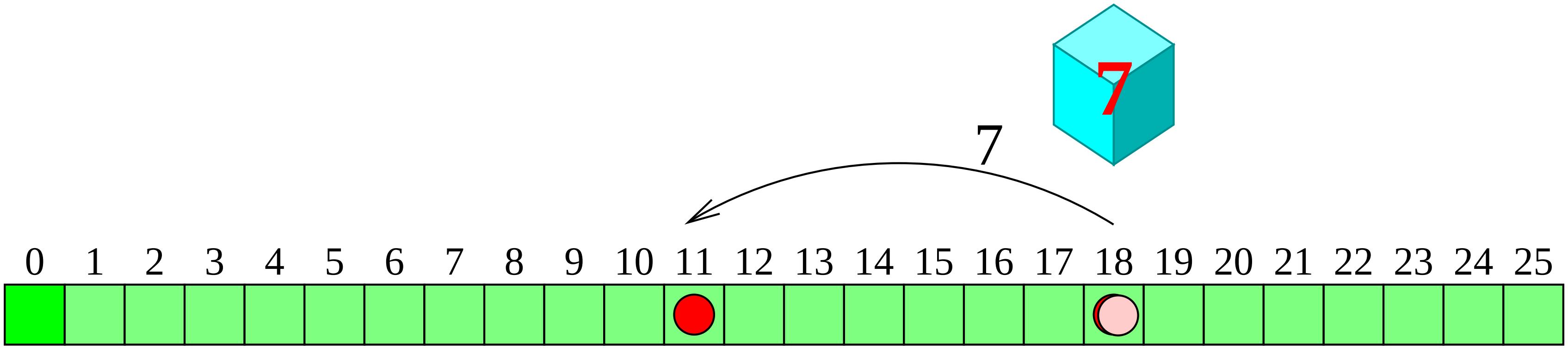

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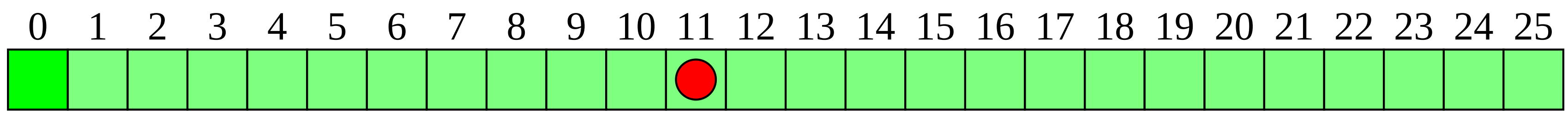

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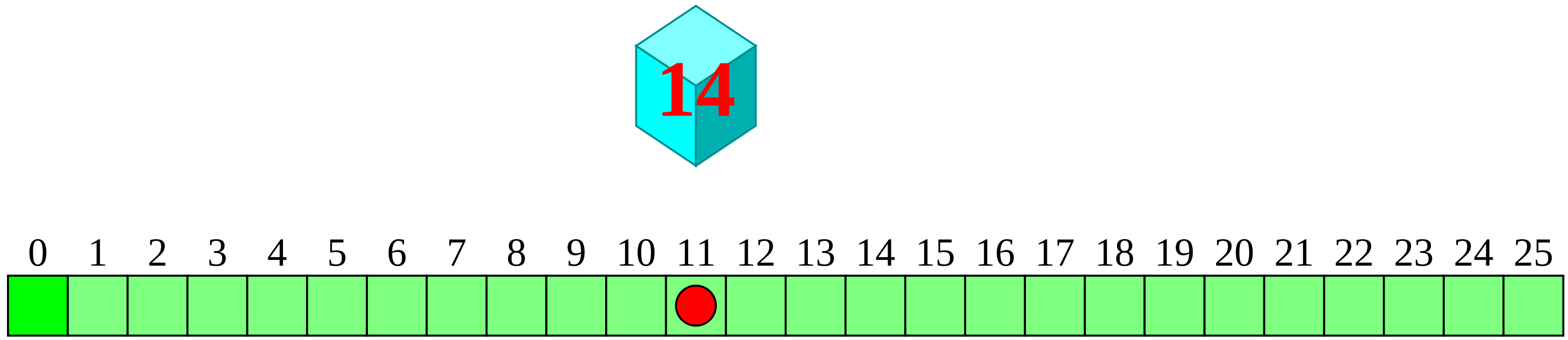

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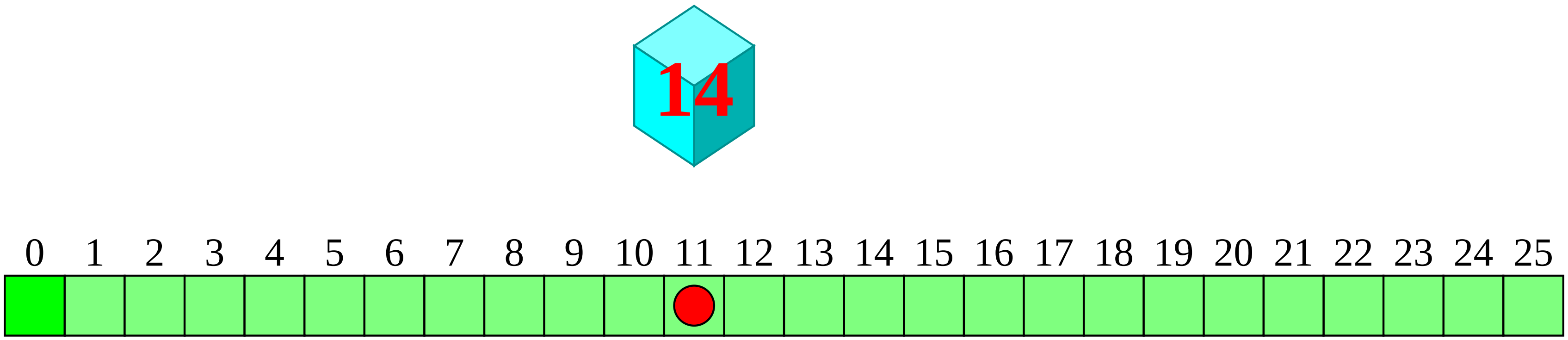

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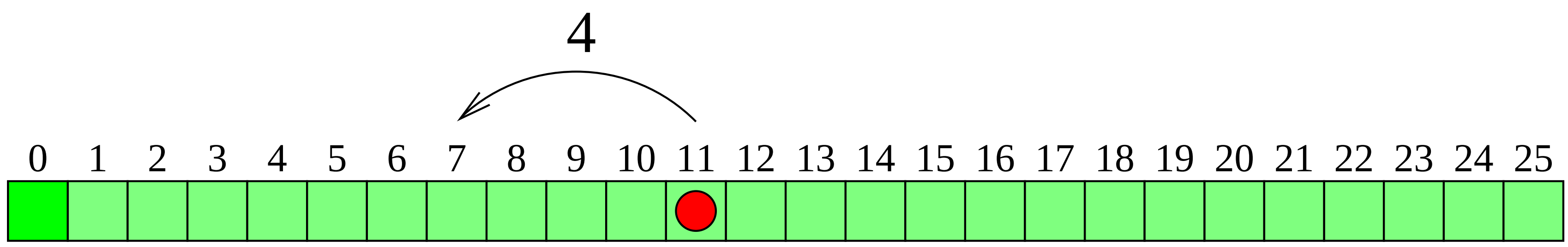

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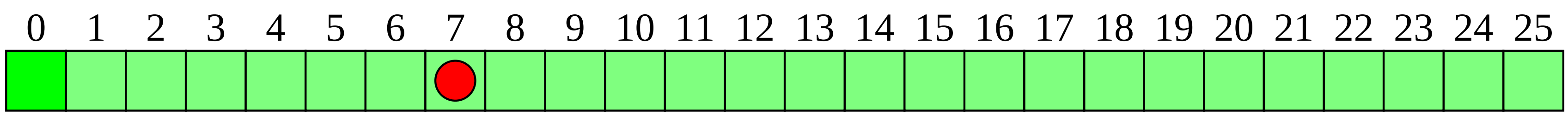

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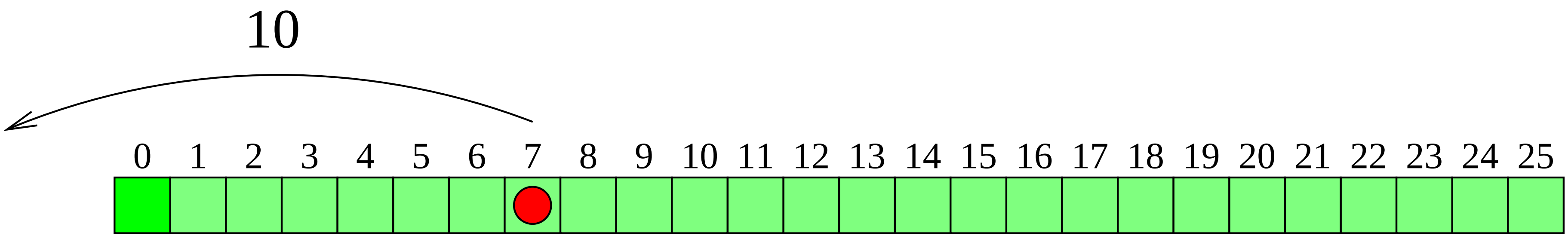

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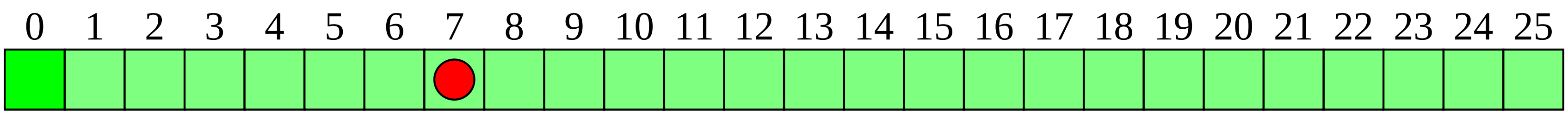

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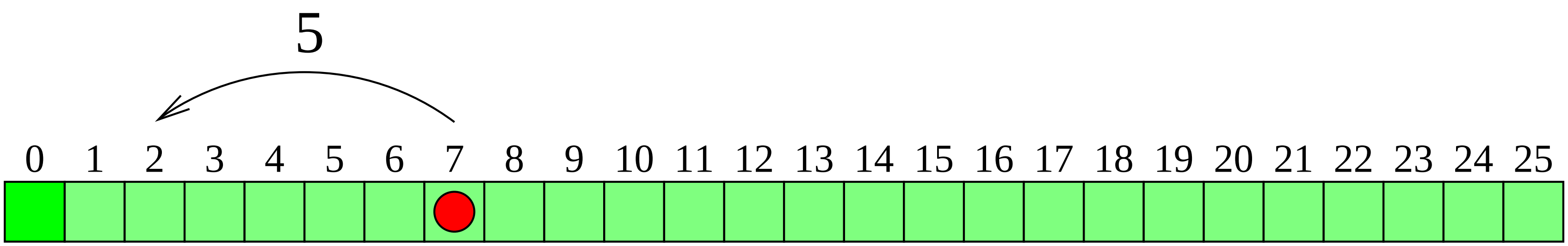

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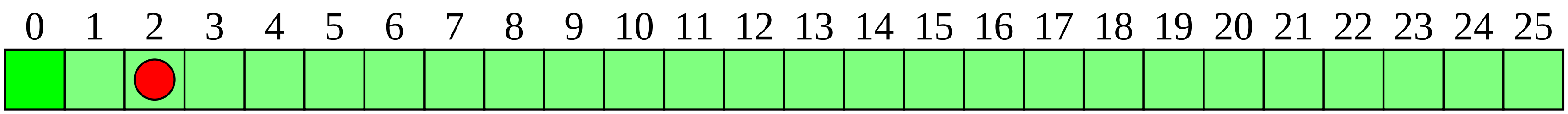

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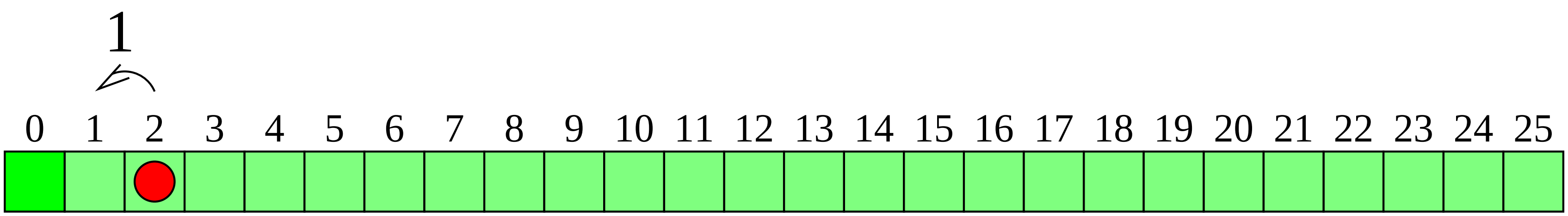

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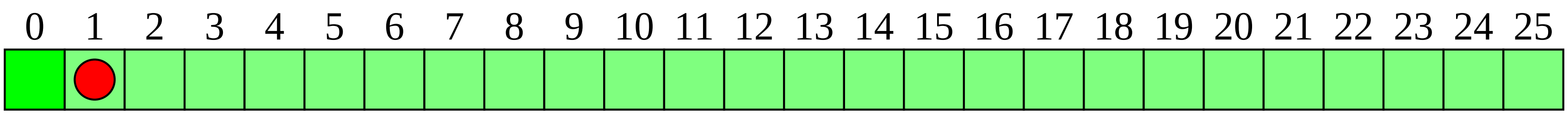

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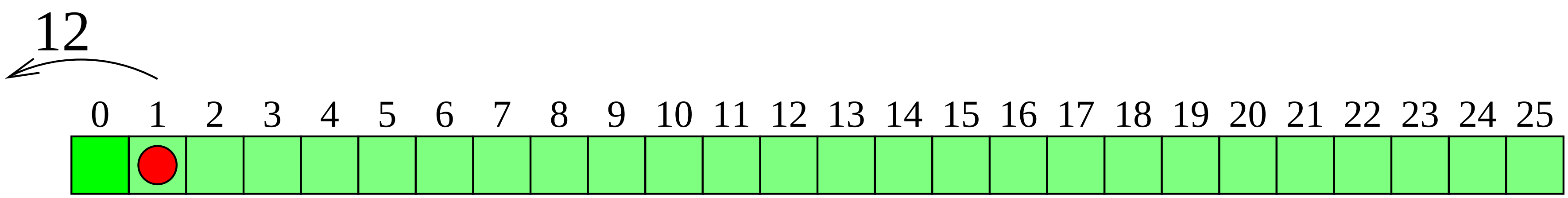

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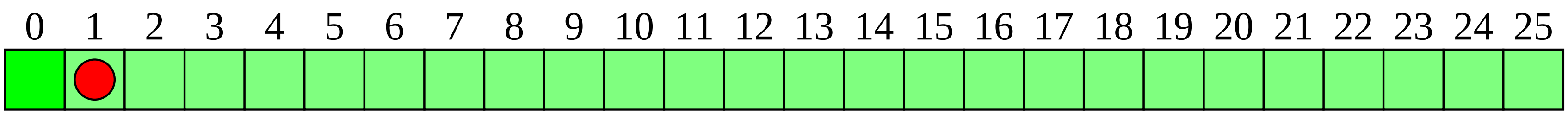

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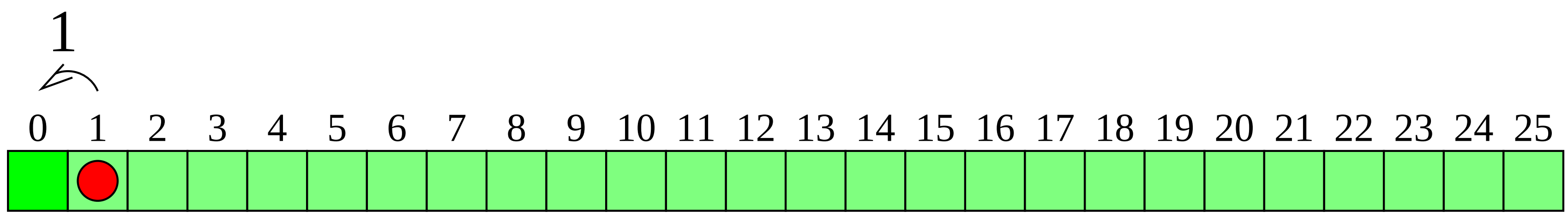

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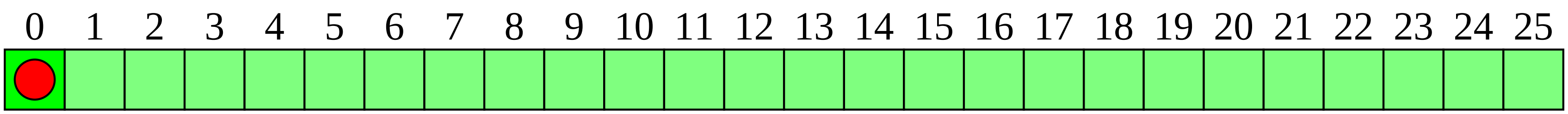

Done!

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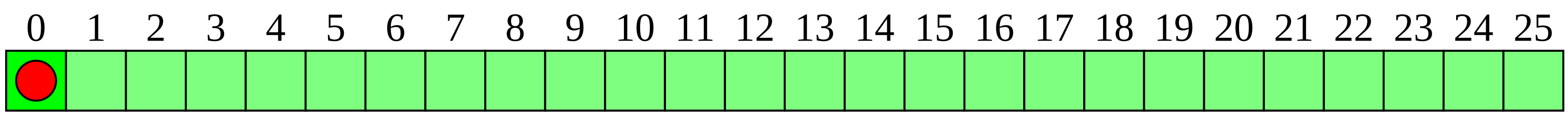

Aim: Get to 0 fast.

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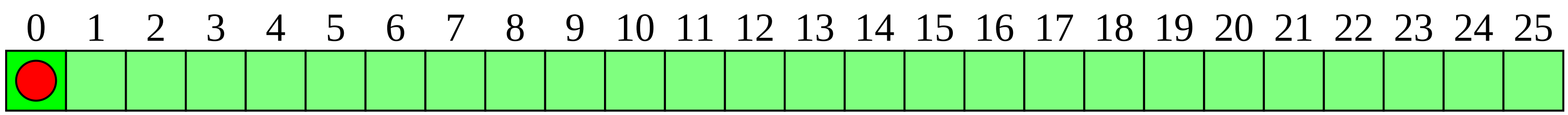

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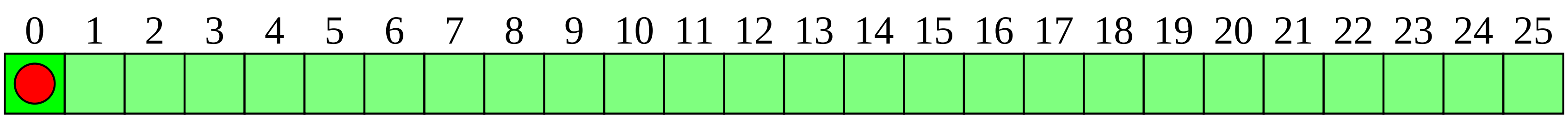

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$$\boxed{\text{What is } \mathbf{E}(T) = \mathbf{E}_\mu(T)?}$$

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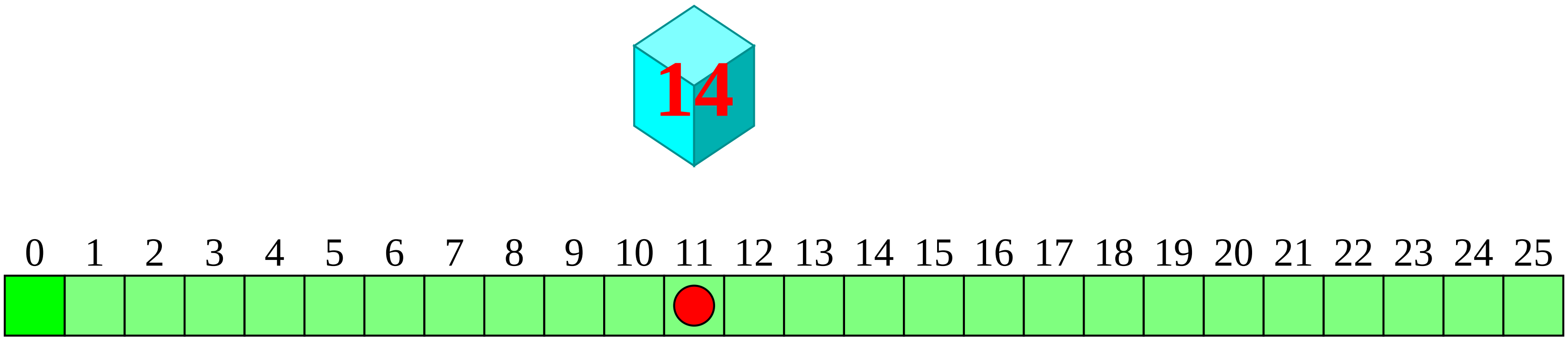

Probability distribution μ on distances: **strategy**.

How small can $\mathbf{E}_\mu(T)$ be? (depending on strategy μ?)

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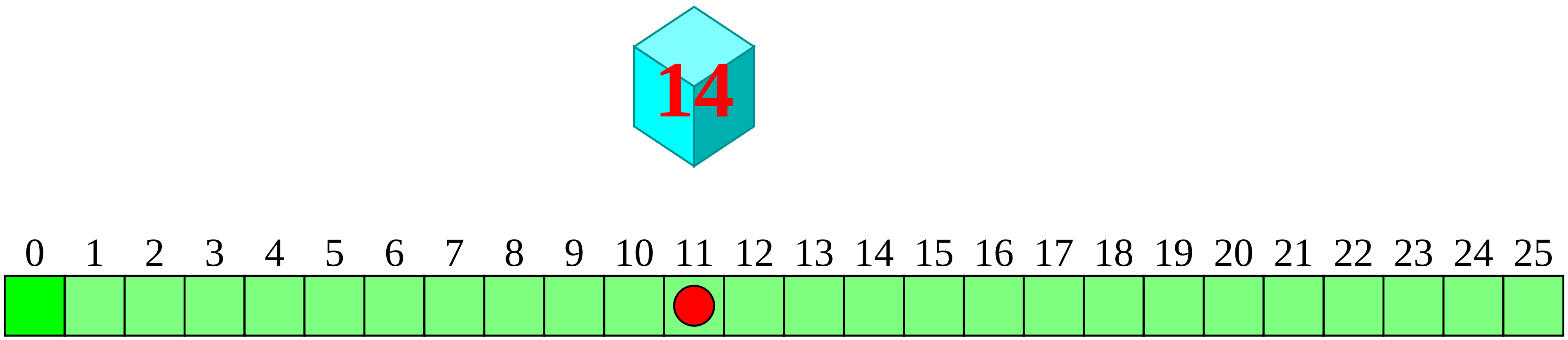

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What do good "**strategies**" $\boldsymbol{\mu}$ look like?

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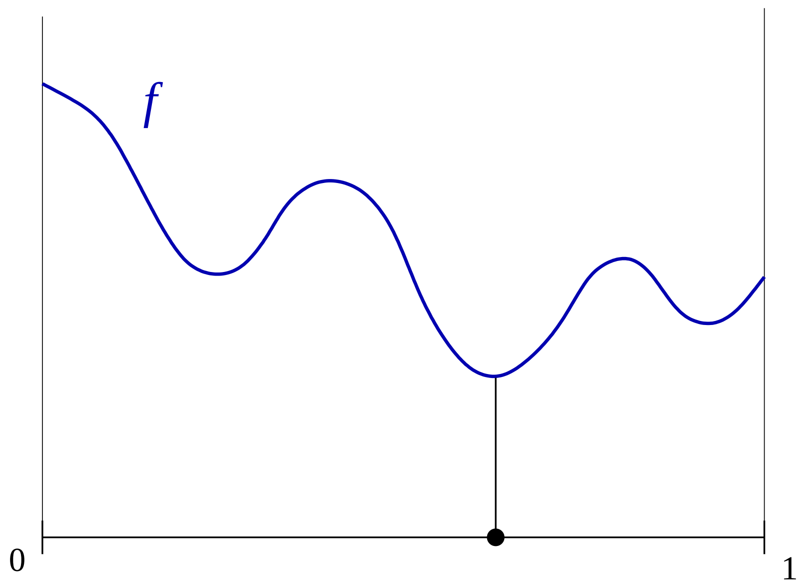

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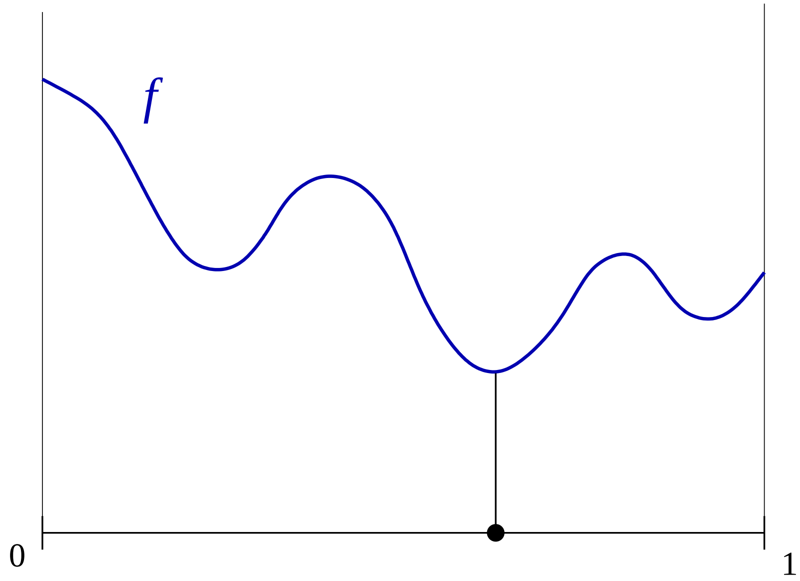

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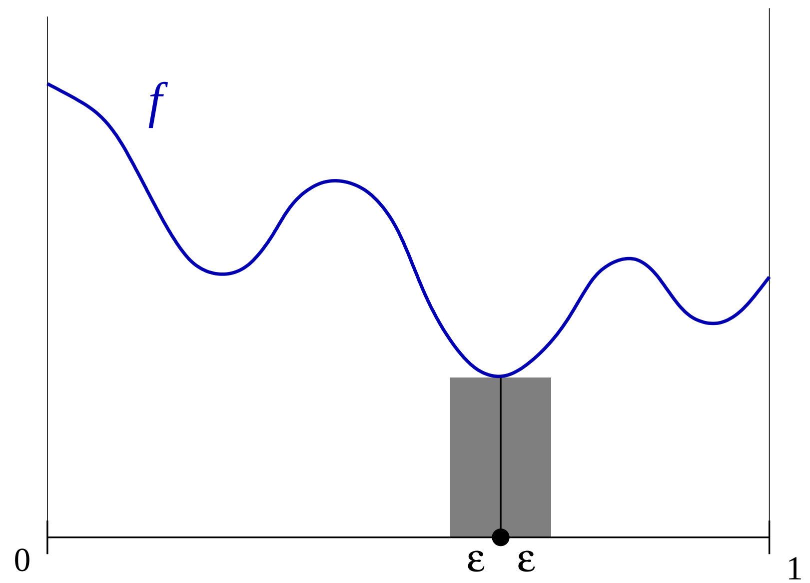

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Heuristic **cannot/does not** use information on what f looks like!

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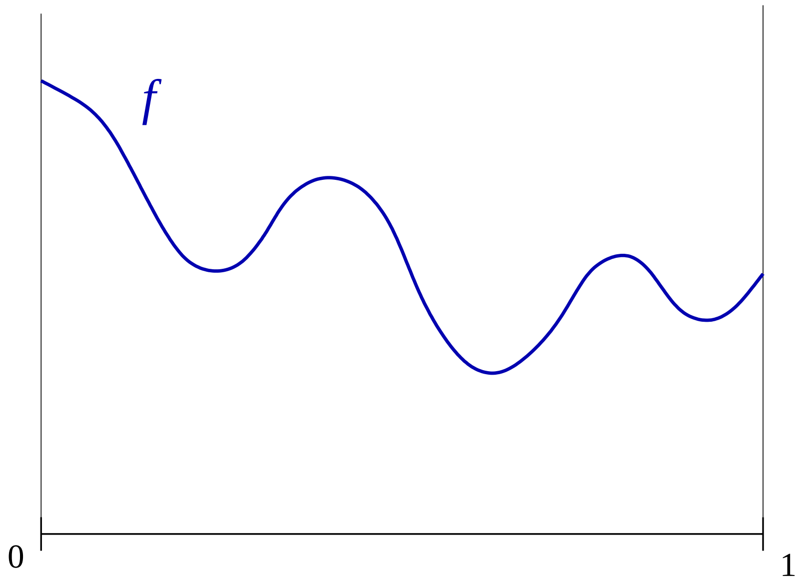

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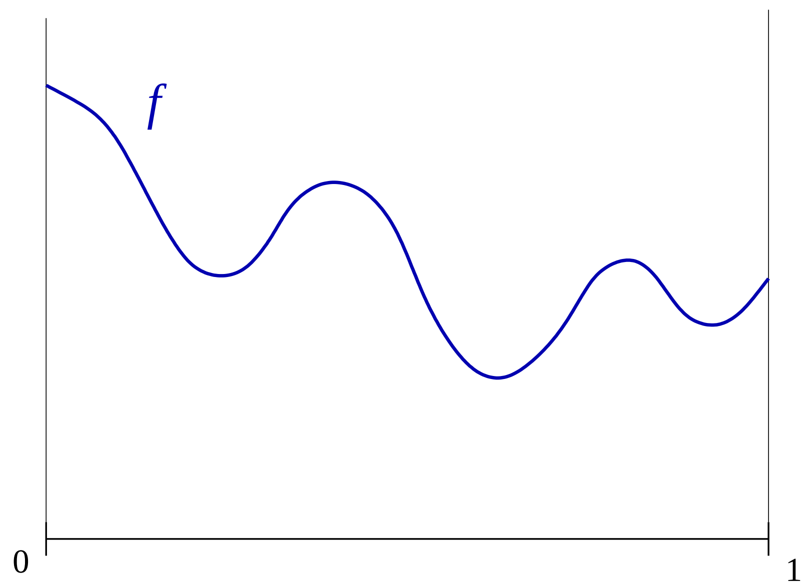

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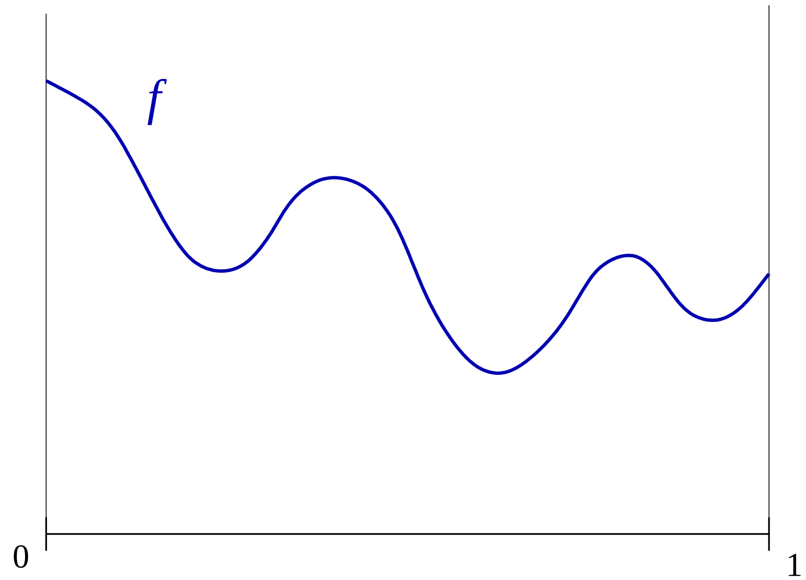

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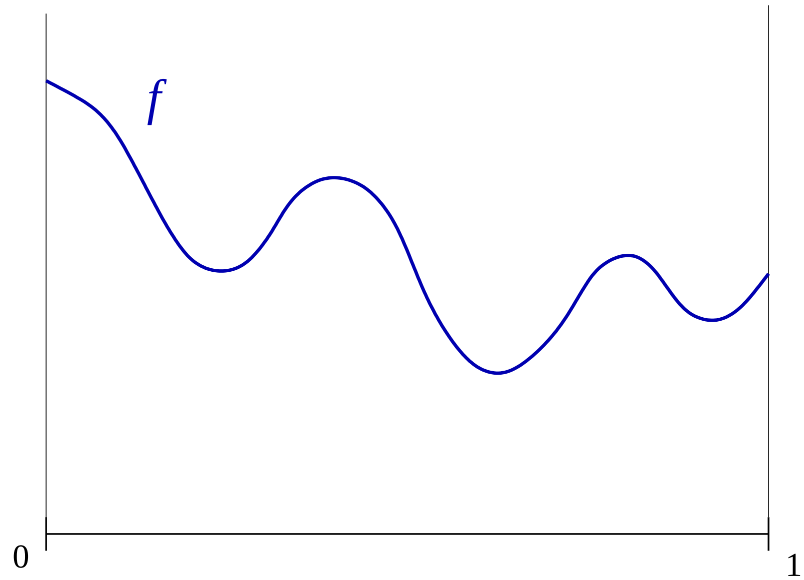

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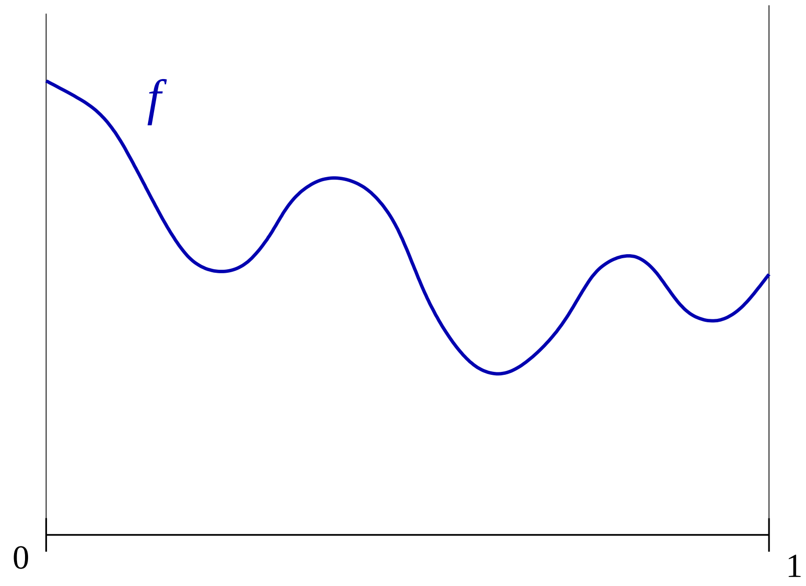

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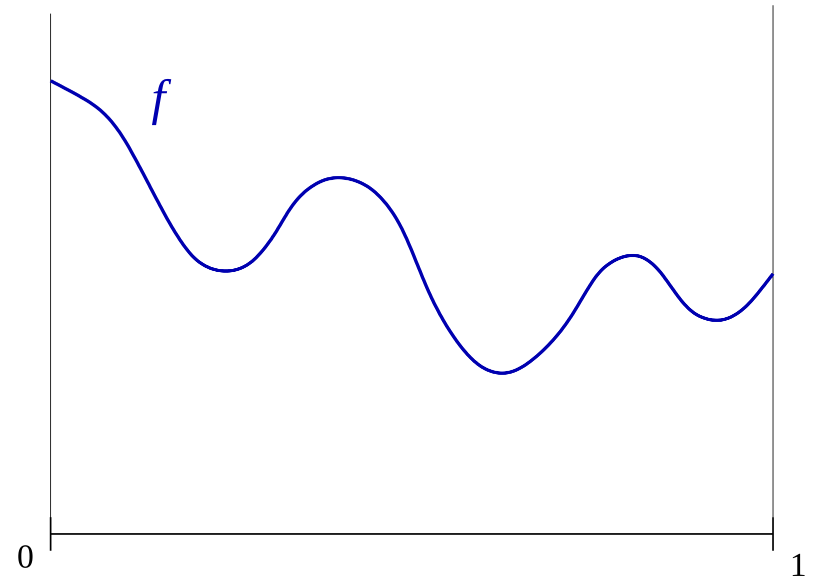

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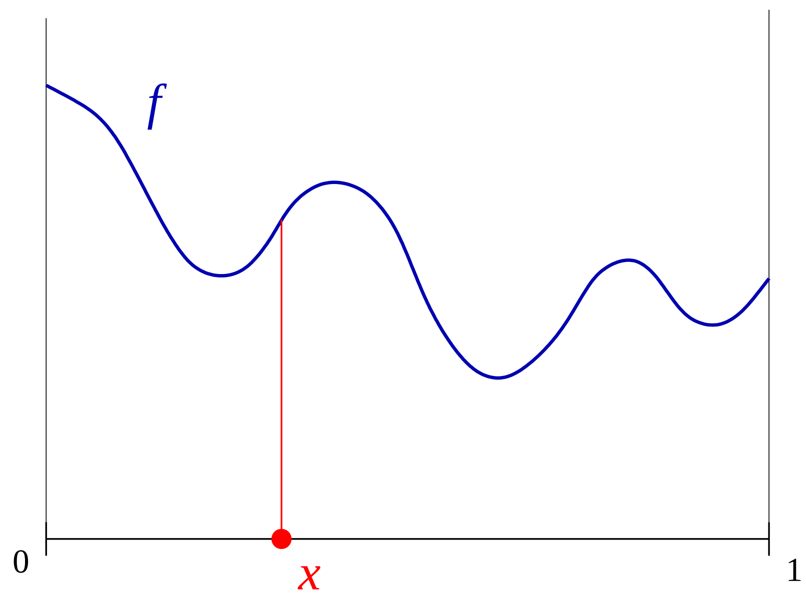

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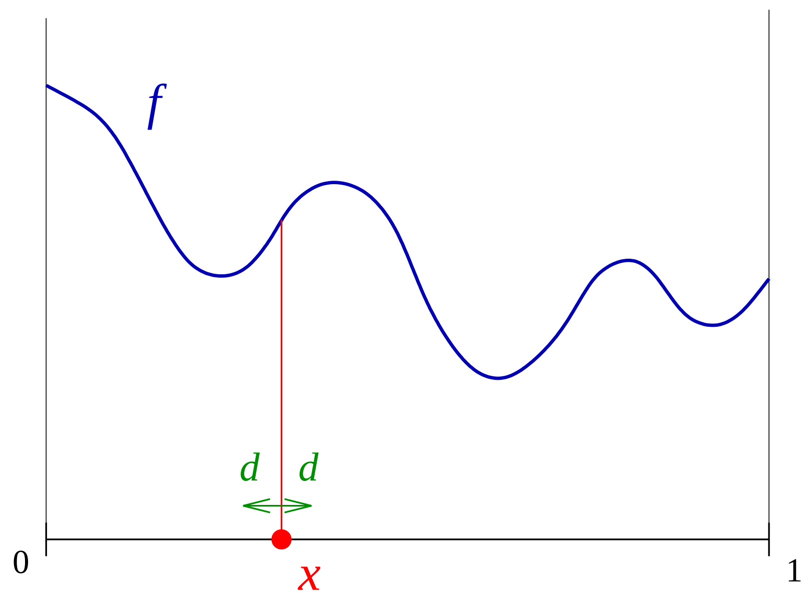

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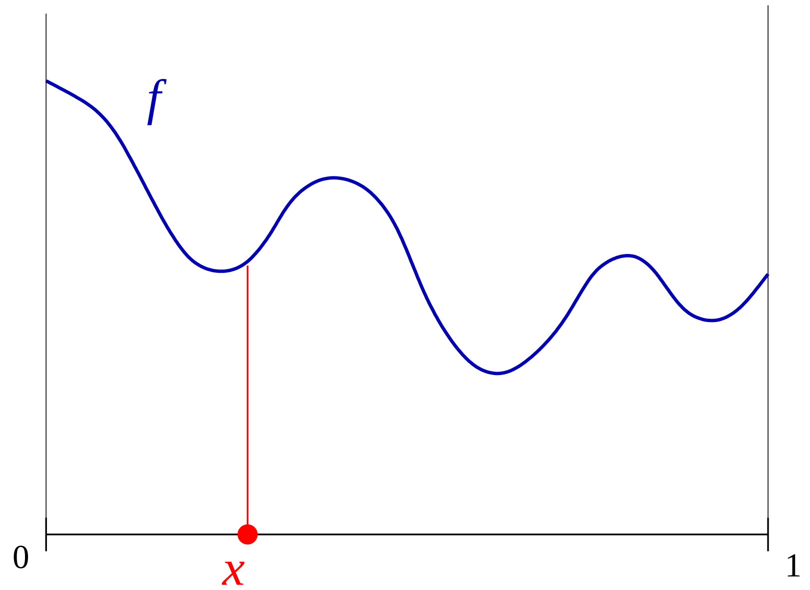

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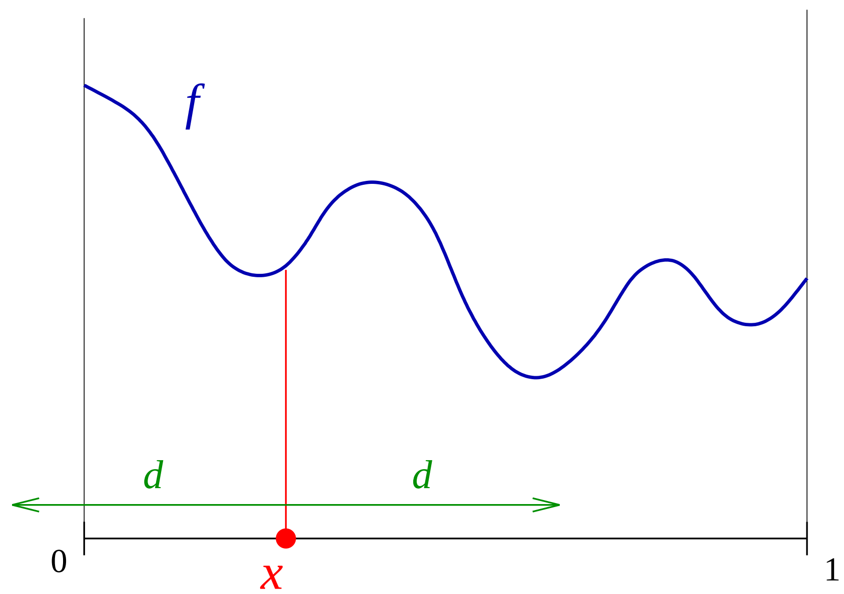

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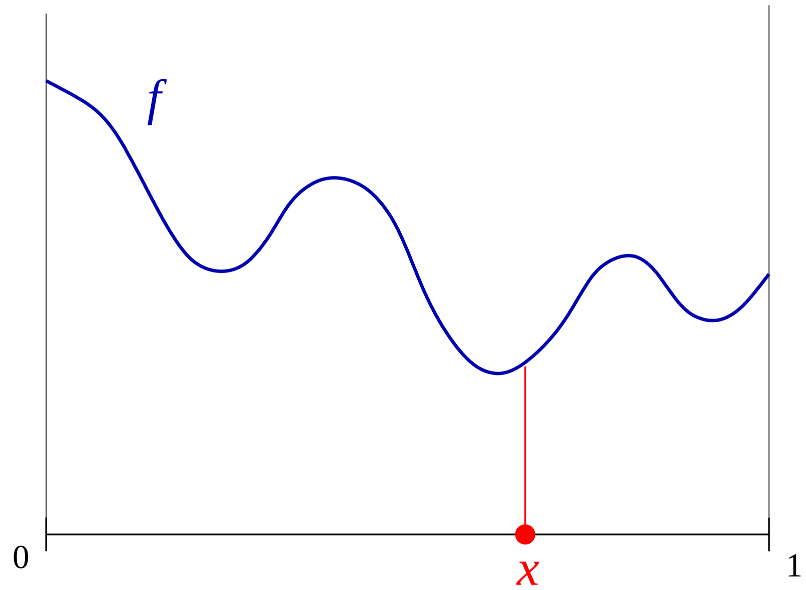

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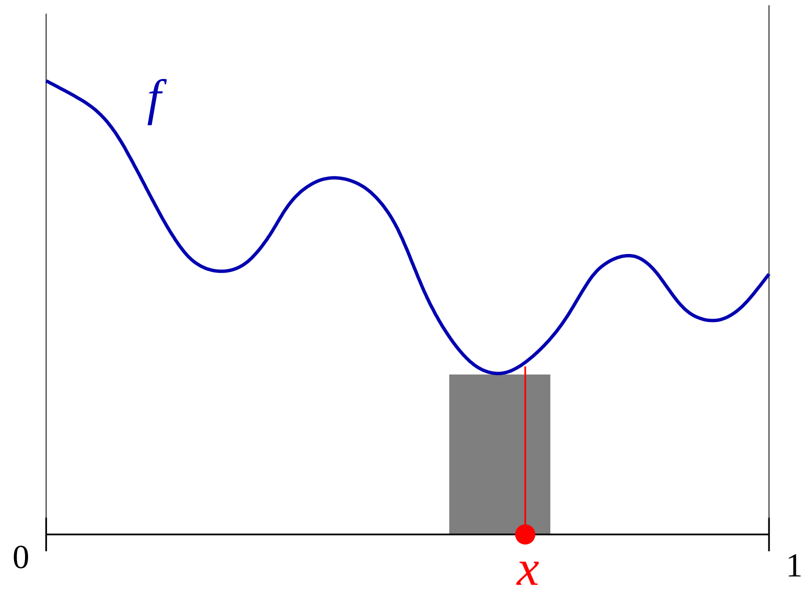

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All orders of magnitude are chosen with the same probability:

"**scale invariant sampling**".

Experiments: good for diverse functions f, possible to escape from local minima . . .

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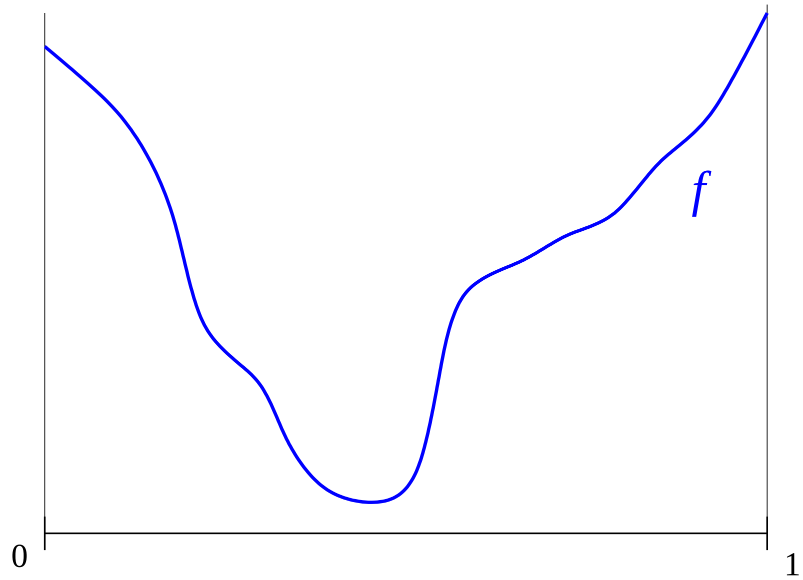
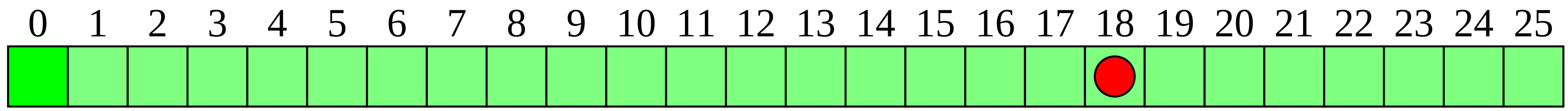
Known: ([Rowe, Hidović 2004]) If f is **unimodal**,



then for the "scale invariant distribution" the expected time for ε-approximating the minimum is $O(\log(1/\varepsilon)^2)$.

Q: Can any fixed nonadaptive strategy ($=$ distribution) be better on unimodal functions?

Our model, our results



Our game:

Discretization! Use $[0, n] = \{0, \ldots, n\}$ instead of $[0, 1]$.

Study (unimodal) function $f \colon [0, n] \ni x \mapsto x$:

Theorem
There are distributions μ with $\mathbf{E}_\mu(T) = O((\log n)^2)$.

Main Theorem
For every distribution μ: $\mathbf{E}_\mu(T) = \Omega((\log n)^2)$.

Implies bound $\Omega((\log(1/\varepsilon)^2))$ for the ε-approximation problem.

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How prove?

Logarithmic scale

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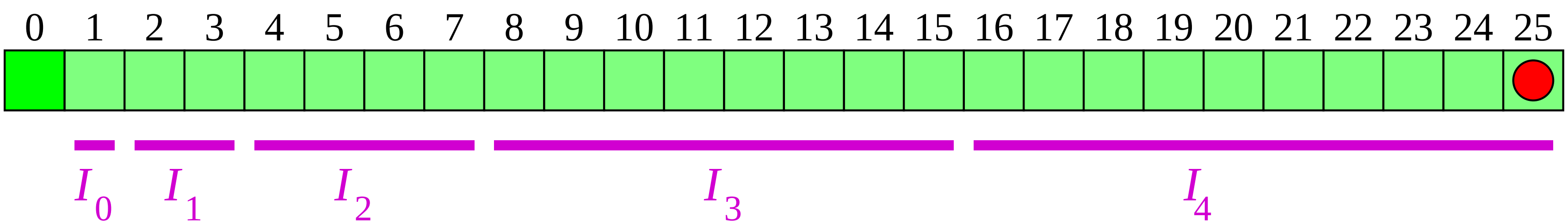

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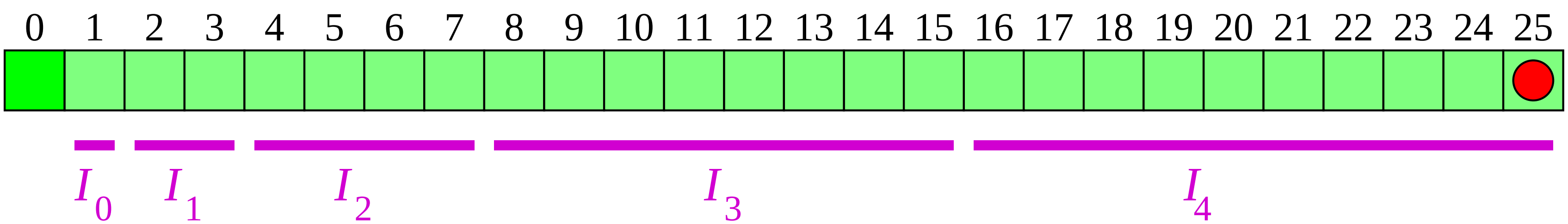

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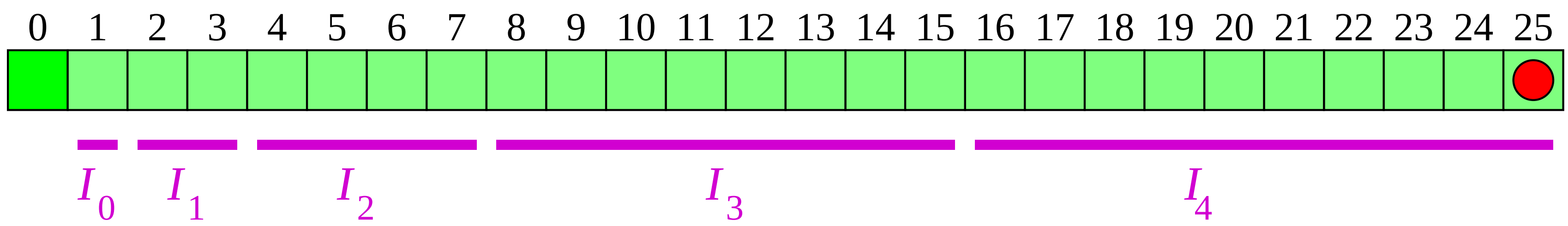

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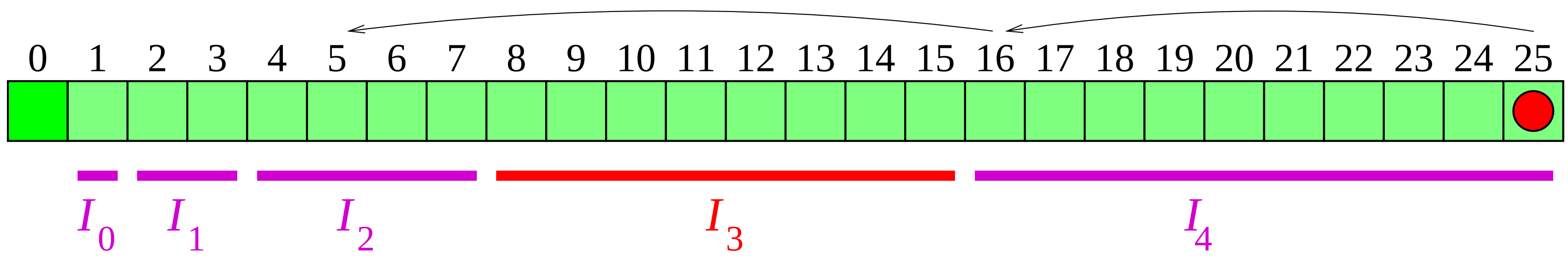

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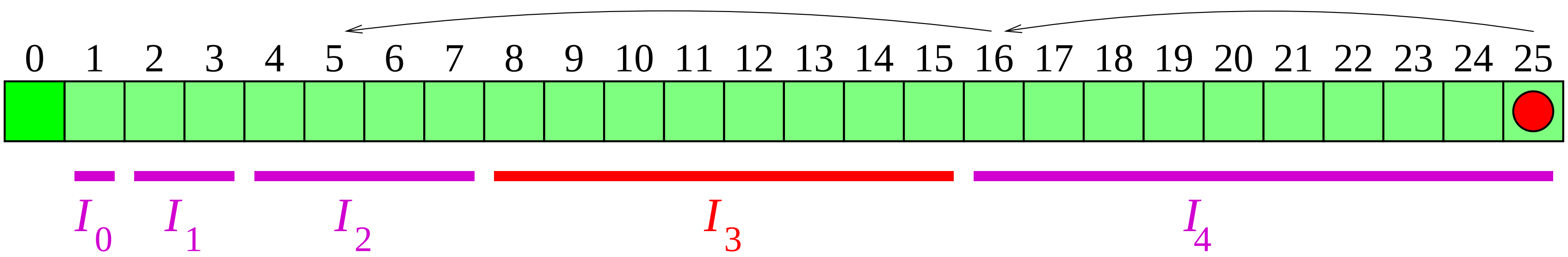

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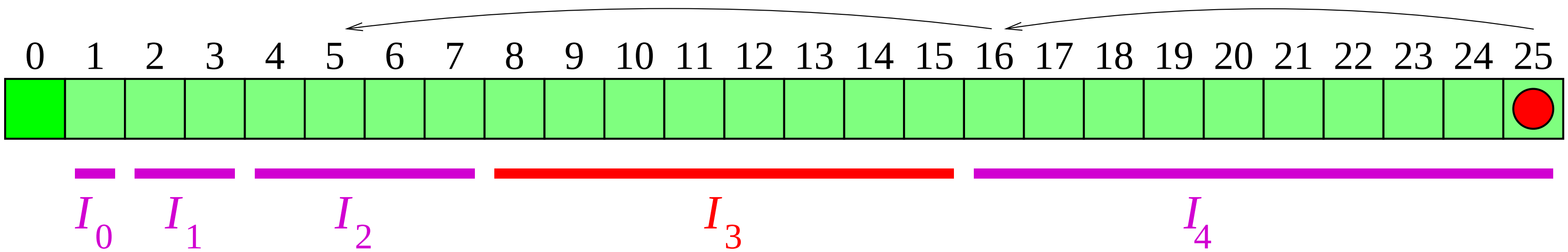

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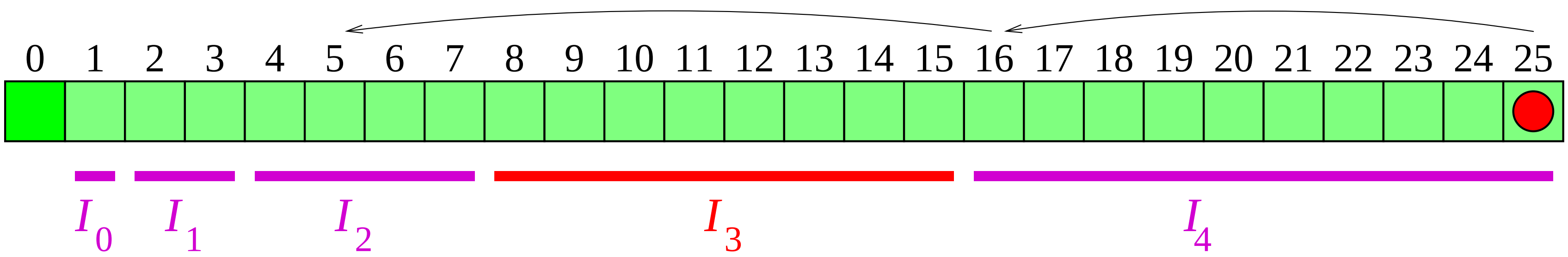

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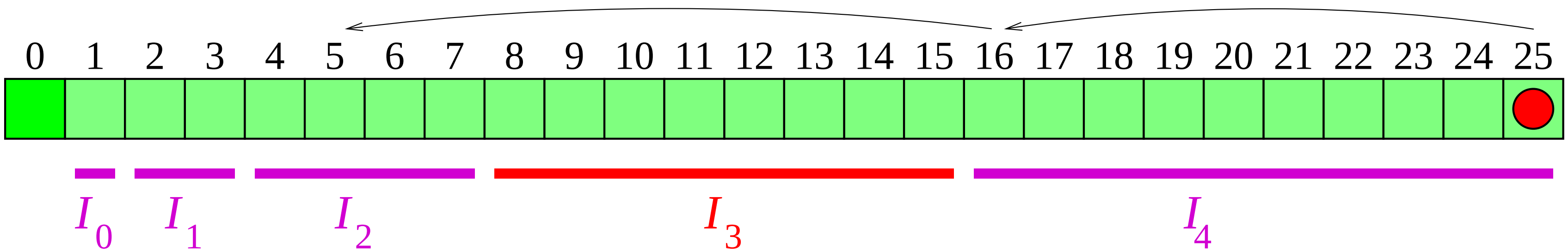

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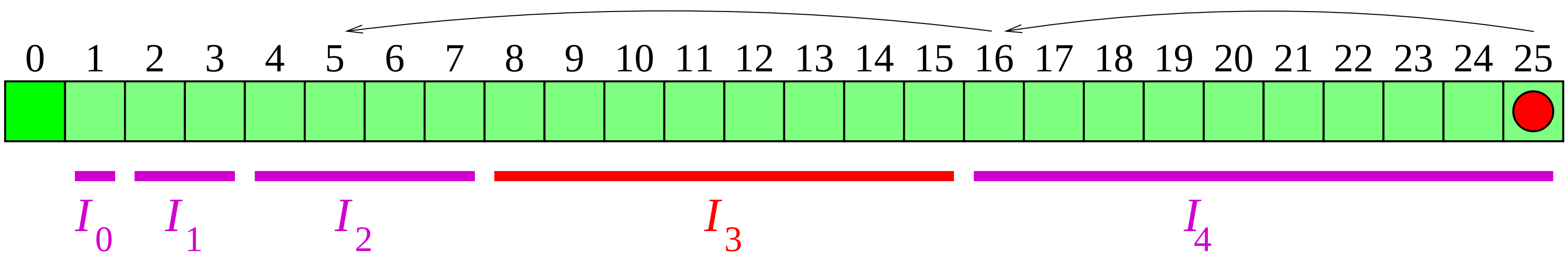

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Main Theorem: "Cannot do better than this."

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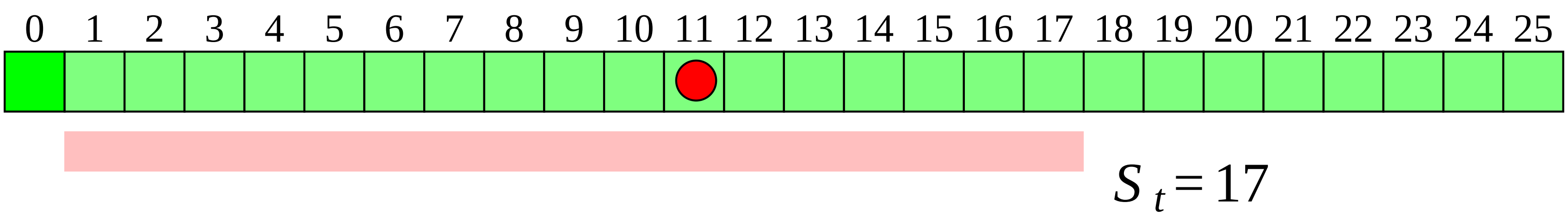

Start: $S_0 = n$.

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Given $\boldsymbol{s = S_{t-1}}$.

Choose $\boldsymbol{d = D_t}$ from $\boldsymbol{[1, n]}$ according to $\boldsymbol{\mu}$.

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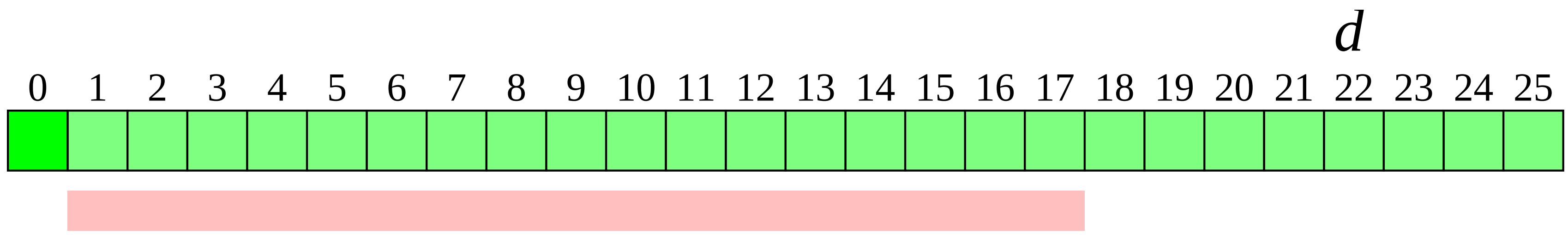

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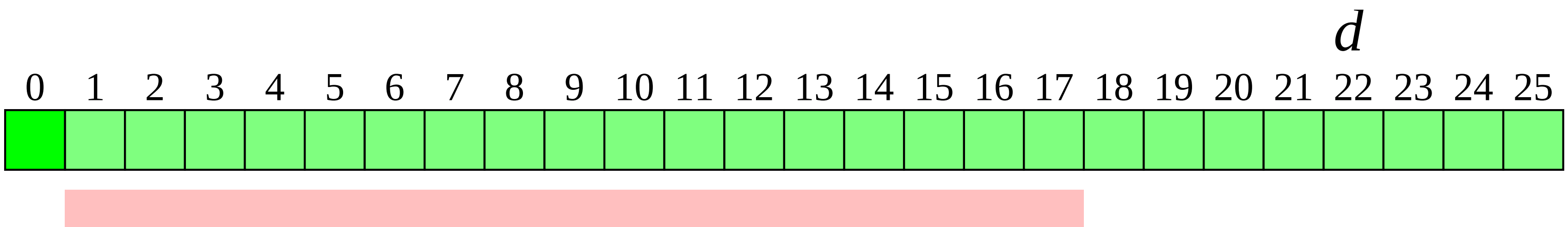

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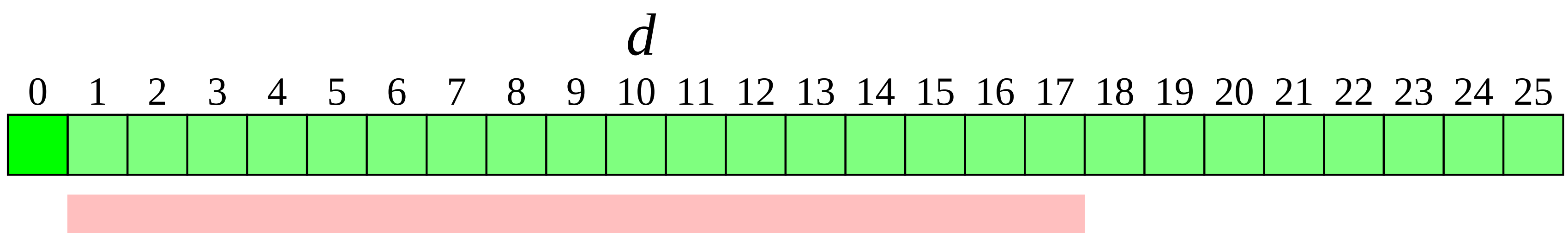

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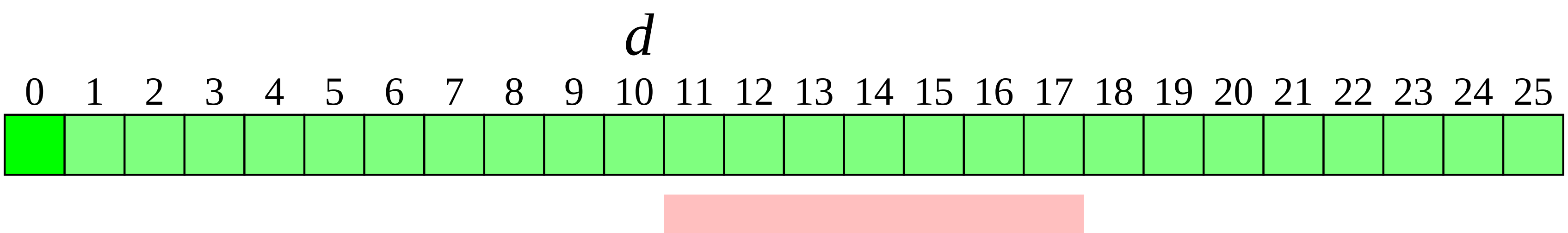

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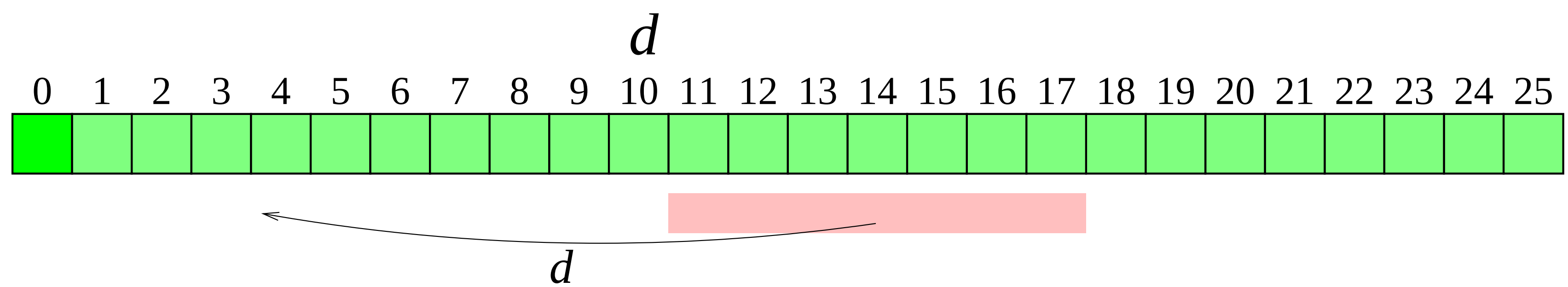

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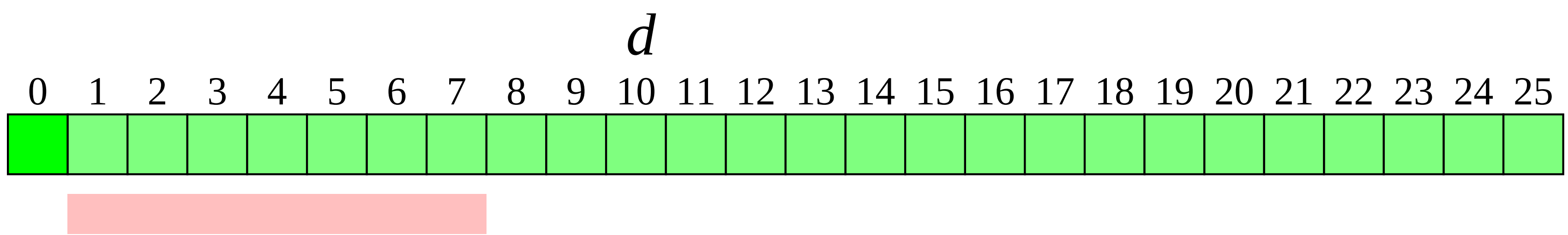

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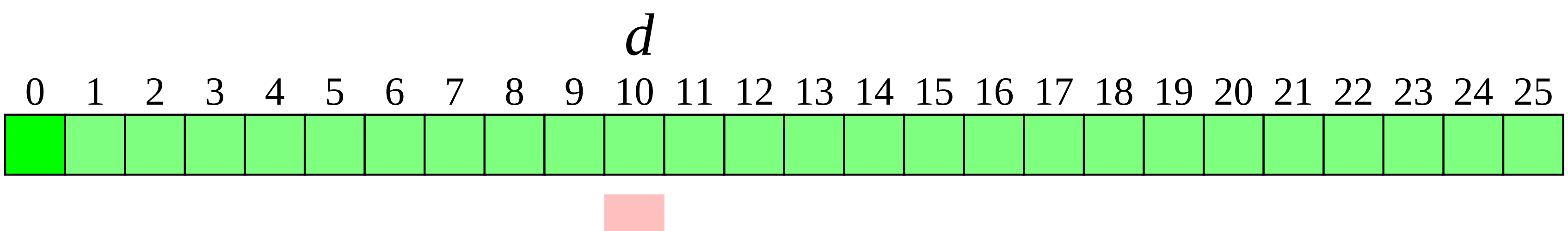

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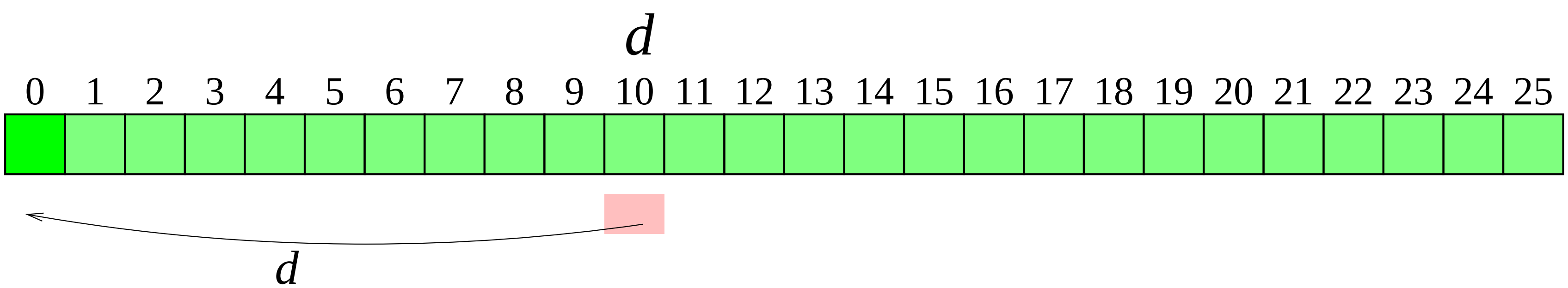

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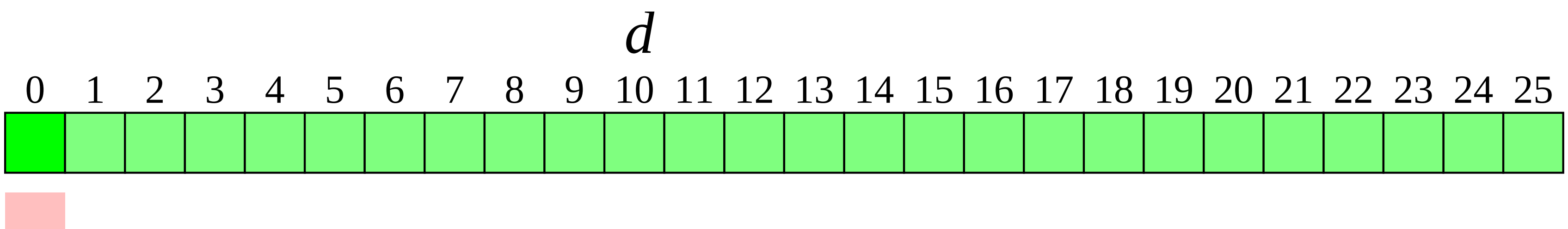

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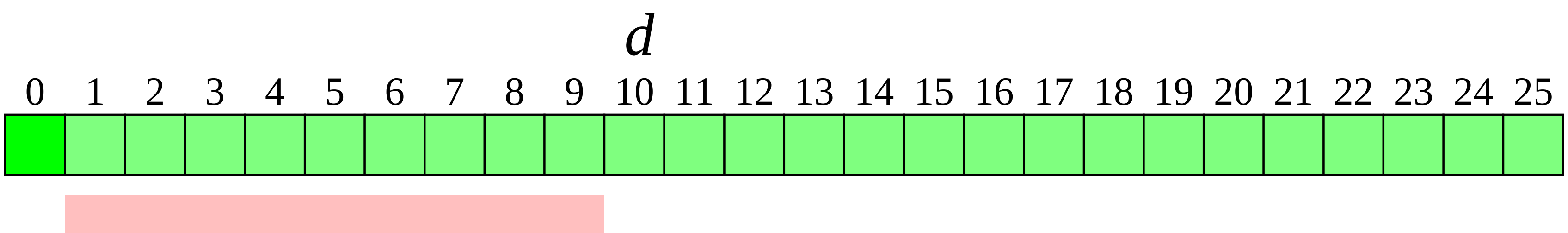

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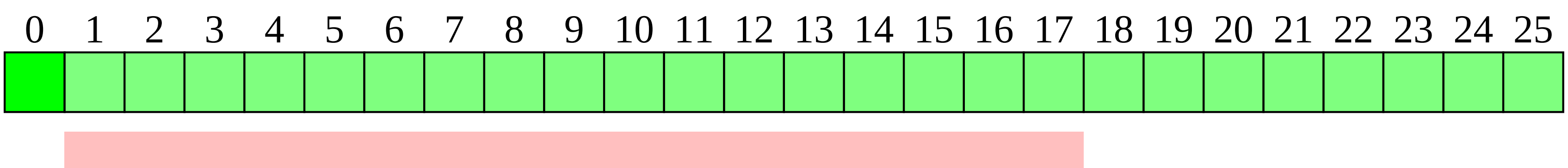

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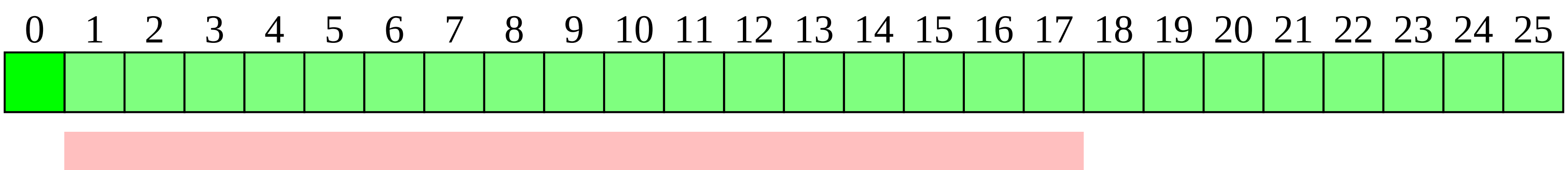

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Denominators add to $O(1)$.

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Intuition: Lower bound for expected cost to get from s to 0.

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Then not too difficult (Version of **Wald's equation**):

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By straightforward calculation!

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On average:

$$\sum_{1 \le d \le s} \frac{\mu(d)}{s} \cdot \sum_{1 \le a \le s} \frac{1}{\sum_{1 \le b \le a} \mu(b)\sqrt{ab} + \sum_{a < b \le n} \mu(b) a^{3/2} \sqrt{b}}$$

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Now $\mu(b)/(\mu(b)\sqrt{ab}) \le \dfrac{\sqrt{s}}{a}$ and . . .

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Similarly for the other cases. □

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Thank you.