Solving Problems with Critical Variables

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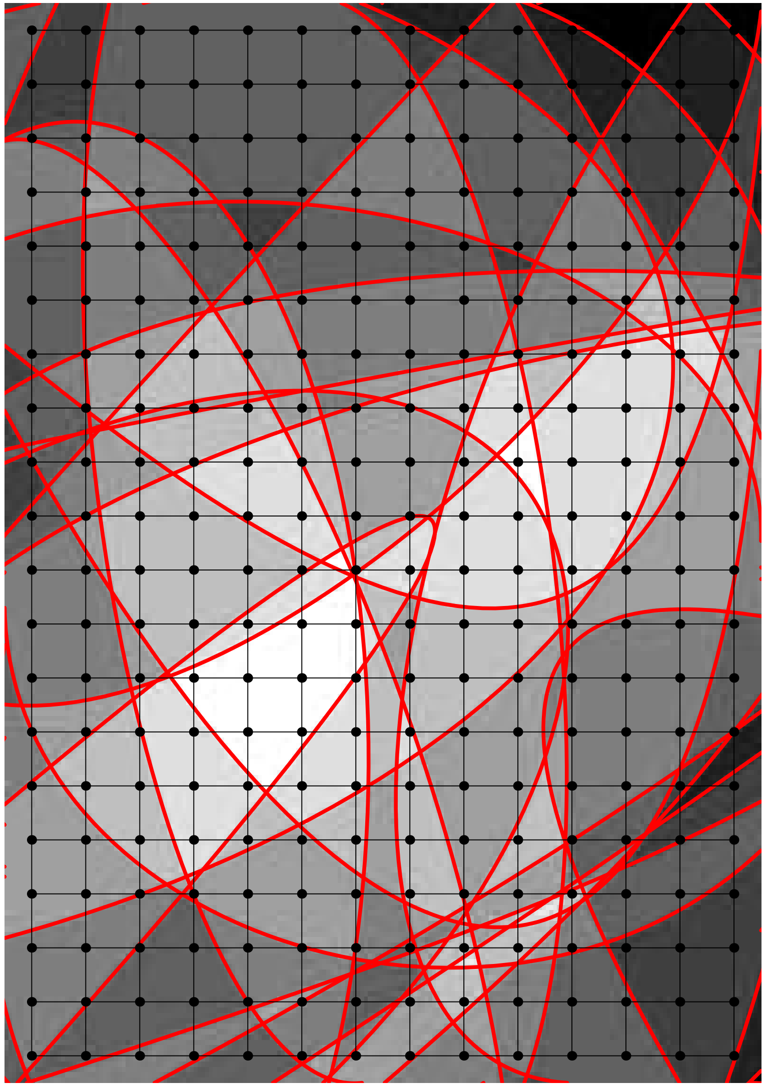
based on last year's GECCO paper

Outline

1. **Critical Variable Problems**
2. Solving Critical Variable Problems
3. A Toy Example
4. Real Optimisation Problems
5. Max-Sat



Critical Variable Problems

- We consider a *special* class of problems—the critical variable problems—with the following properties

- The problem is represented by a string of variables divided into two groups

  CBBCEAAEAAAAACBDABBBADEAEBCDBA

  ⋆ a small set of critical variables

  ⋆ all other variable are normal variables

- The position of the critical variables are unknown

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Normal and Critical Variables

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- If the critical variables are fixed then it is easy to optimise the normal variables, e.g. by a hill-climber
- The optimal values for the normal variables will depend on the critical variables
- The critical variables are (at least approximately) uncoupled
- Because the normal variables depend on the critical variables we cannot hill-climb to optimise the critical variables

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Landscape of Critical Variable Problems

- Every configuration of the critical variables is associated with a local optima (found by performing hill-climbing on the normal variables)

- The local minima can be some distance apart since changing a critical variable might require many normal variables to be changed

- The more good critical variables the better the local optima (since the critical variables are not strongly coupled)

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The Challenge of Critical Variable Problems

- We have a problem with a very large number of local optima
- We don't know which variables are the critical variables
- The probability of guessing which variables are critical variables is vanishingly small—we are looking for a small number of variables hidden in a crowd of normal variables
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Solving Critical Variable Problems

- We can solve Critical Variable Problems using two levels of search
  - ⋆ use a hill-climber to optimise the normal variables
  - ⋆ use crossover to optimise the critical variables
- We need to optimise the normal variables to evaluate the fitness of the critical variables

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A Hybrid GA Solver

- This leads to a hybrid GA algorithm:

  1. generate a random population
  2. perform hill-climbing on each member of the population
  3. **for** $t = 1$ **to** T

     (a) select the fitter members of the population

     (b) perform uniform crossover on the new population

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What's Happening

- Hill-climbing optimises the normal variables which allows us to evaluate the fitness of the critical variables
- Selection increases the proportion of good critical variables
- Crossover mixes up the critical variables thus allowing exploration of the space of critical variables
- Crossover will mess up the normal variables so that they will need to be re-optimised at each step

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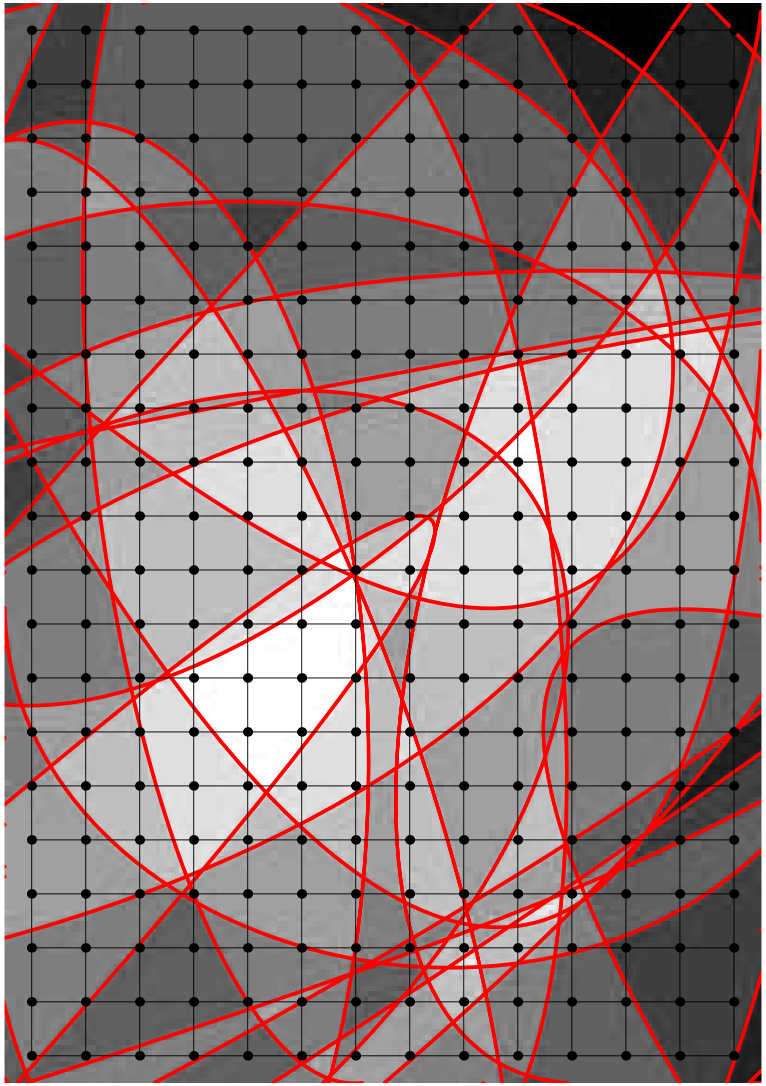

Simple Example

- We consider binary strings with variables $X_i \in \{-1, 1\}$
- Fitness made up of two parts:

$$F = F_{\text{critical}} + F_{\text{normal}}$$

⋆ A contribution depending only on the critical variables

$$F_{\text{critical}} = \sum_{i \in \text{critical}} X_i$$

⋆ The normal variables contribute in blocks

$$F_{\text{normal}} = \sum_{i \in \text{critical}} F_i^{\text{block}}, \qquad F_i^{\text{block}} = X_i \sum_{j \in \text{block}(i)} X_{j_i}$$

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Fitness Landscape

Binary String

																								F
1	-1	1	1	1	-1	-1	1	-1	1	-1	1	-1	1	1	1	1	1	-1	1	-1	-1	-1	1	-2

Fitness Landscape

Critical Variables

																								F
1	-1	1	1	1	-1	-1	1	-1	1	-1	1	-1	1	1	1	1	1	-1	1	-1	-1	-1	1	-2

Fitness Landscape

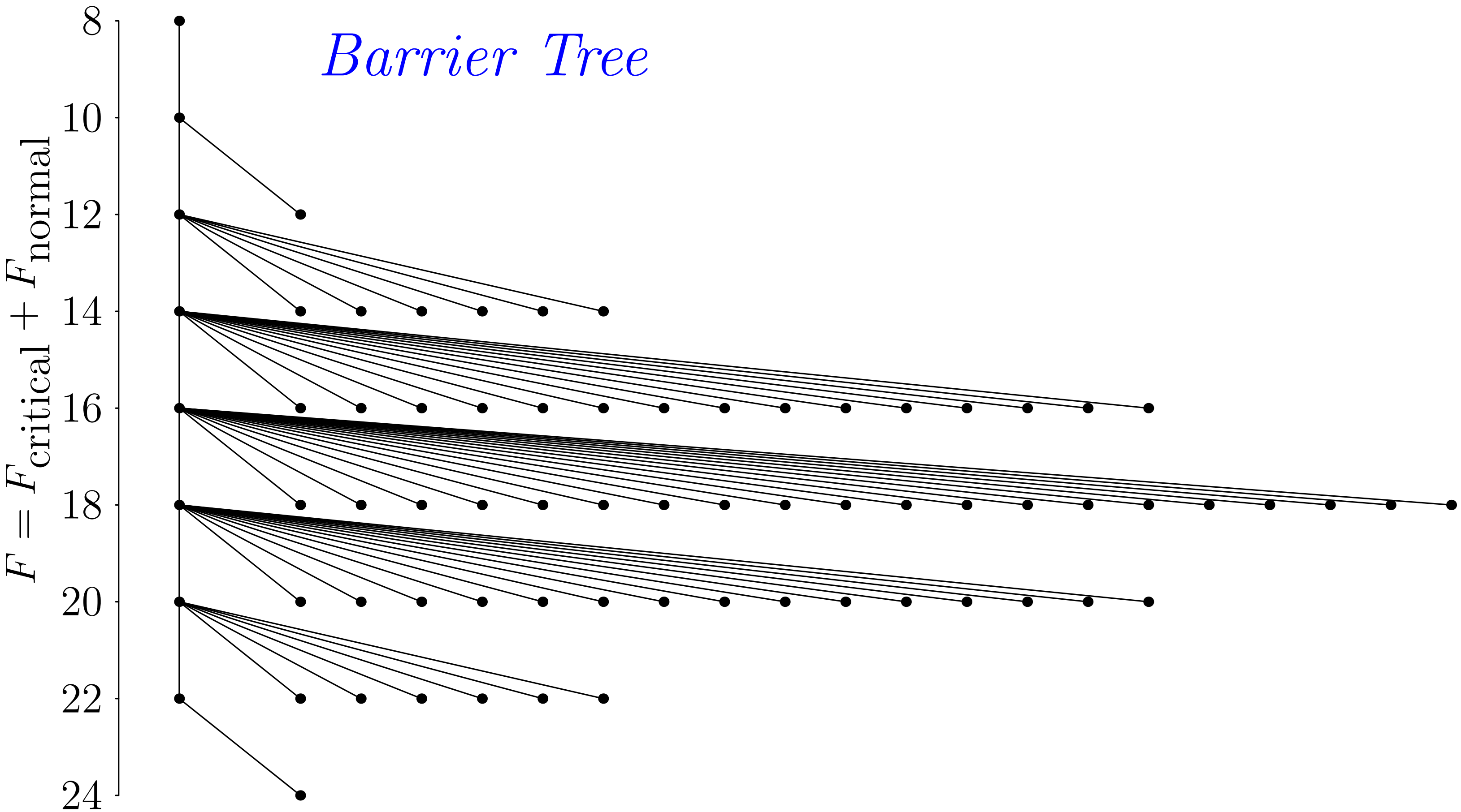
Normal Variables

																								F
1	-1	1	1	1	-1	-1	1	-1	1	-1	1	-1	1	1	1	1	1	-1	1	-1	-1	-1	1	-2

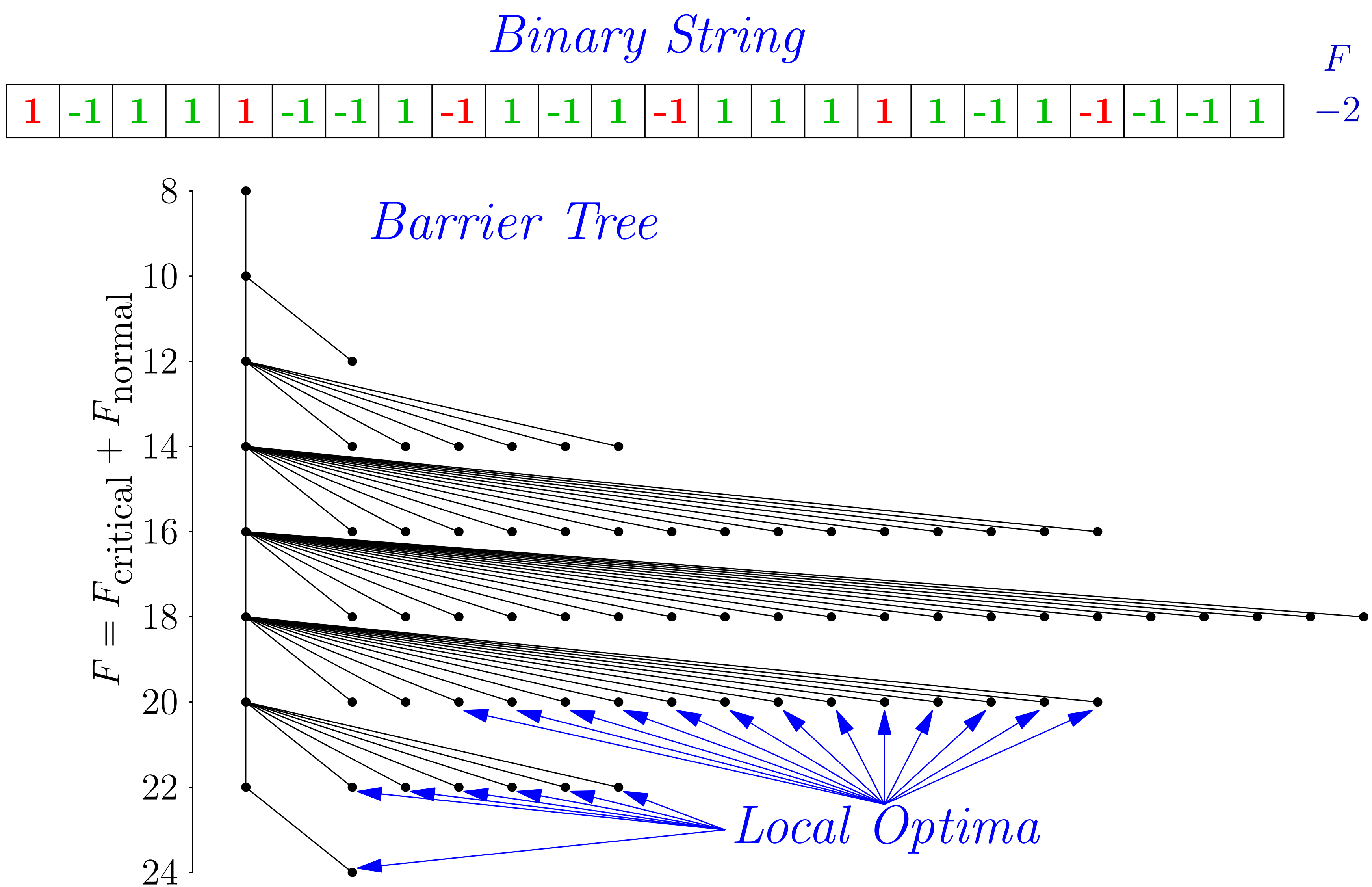
Fitness Landscape

Binary String

1	-1	1	1	1	-1	-1	1	-1	1	-1	1	-1	1	1	1	1	1	-1	1	-1	-1	-1	1	F: -2



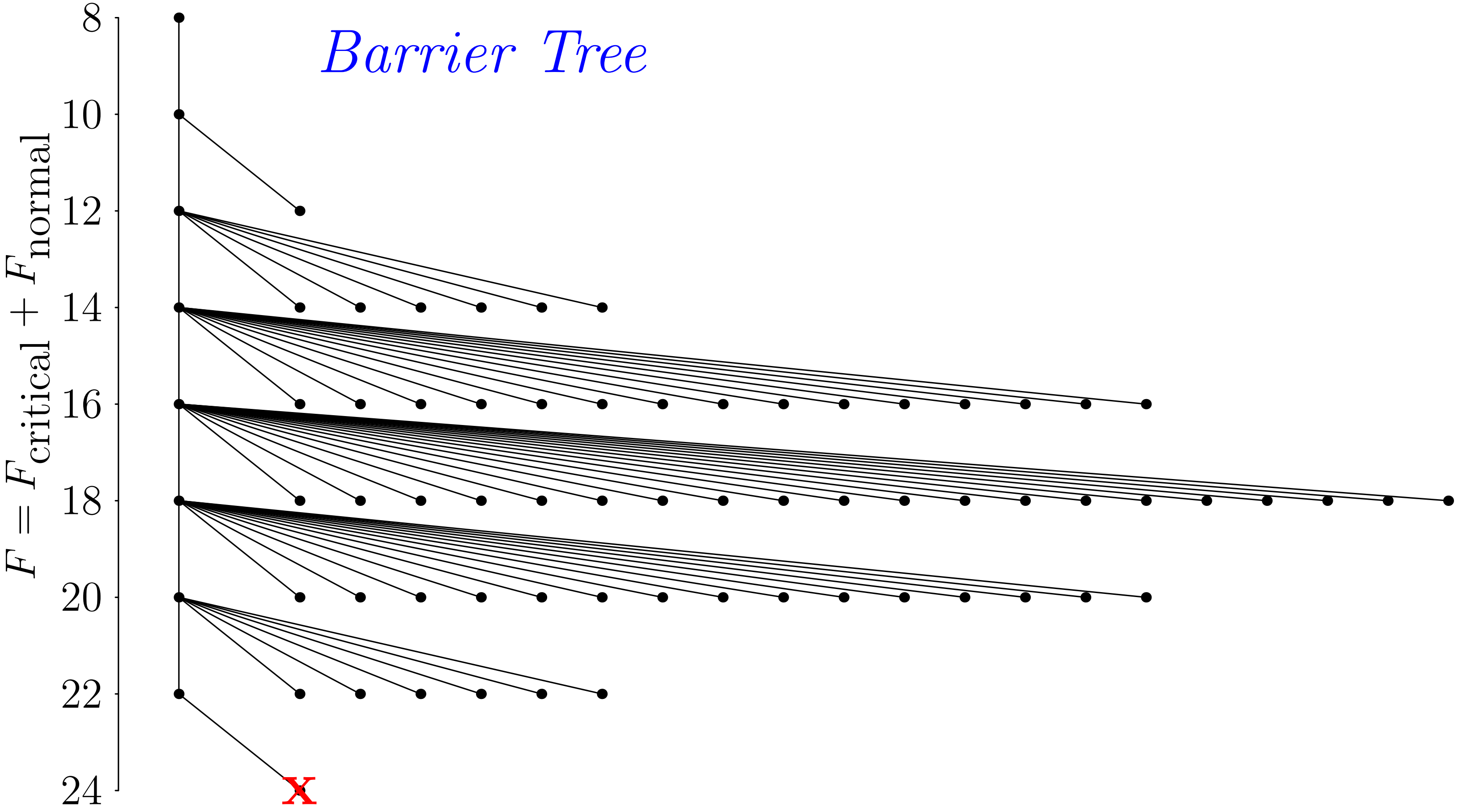
Fitness Landscape



Fitness Landscape

Binary String

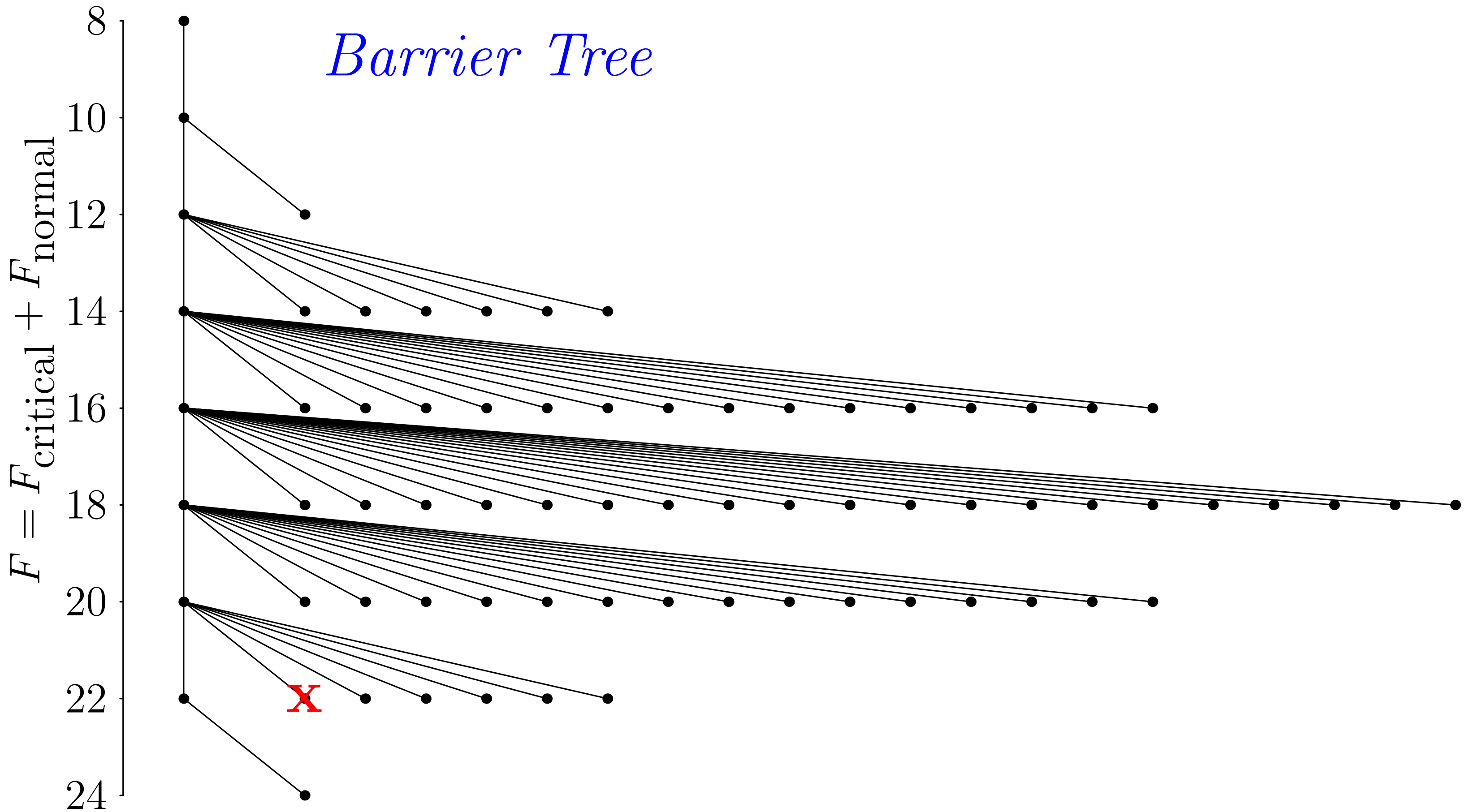
																								F
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	24



Fitness Landscape

Binary String

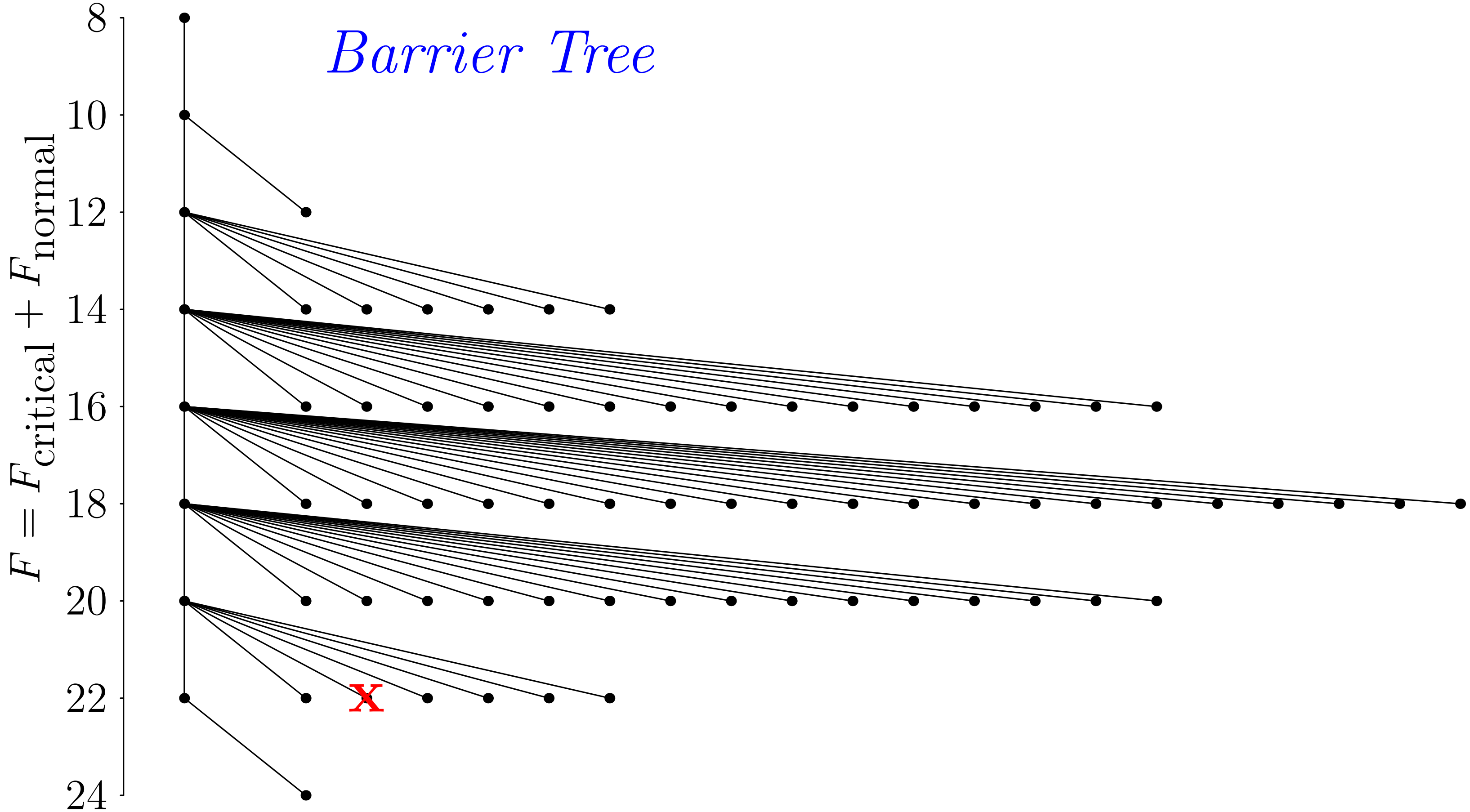
-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	F 22



Fitness Landscape

Binary String

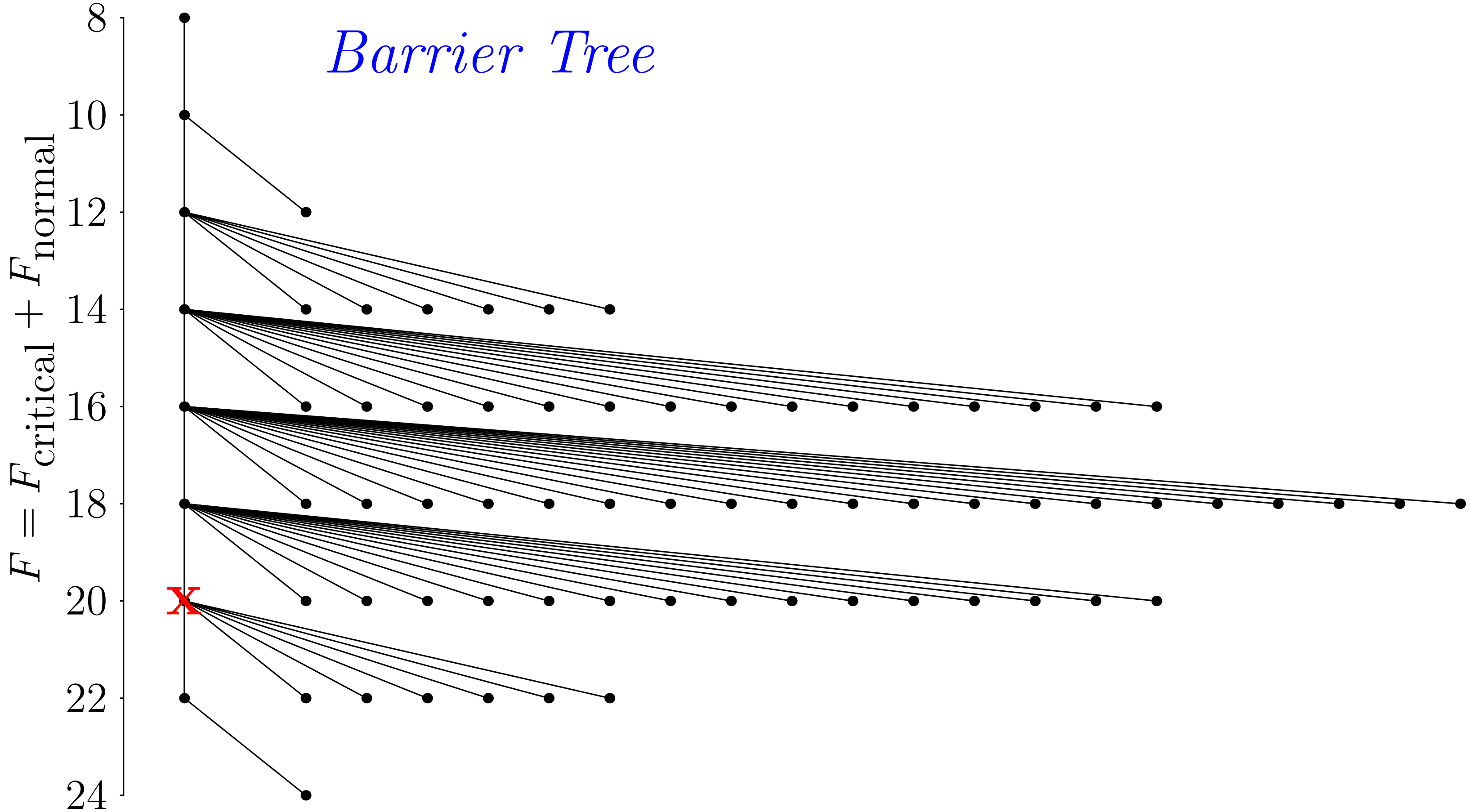
																								F
1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	22



Fitness Landscape

Binary String

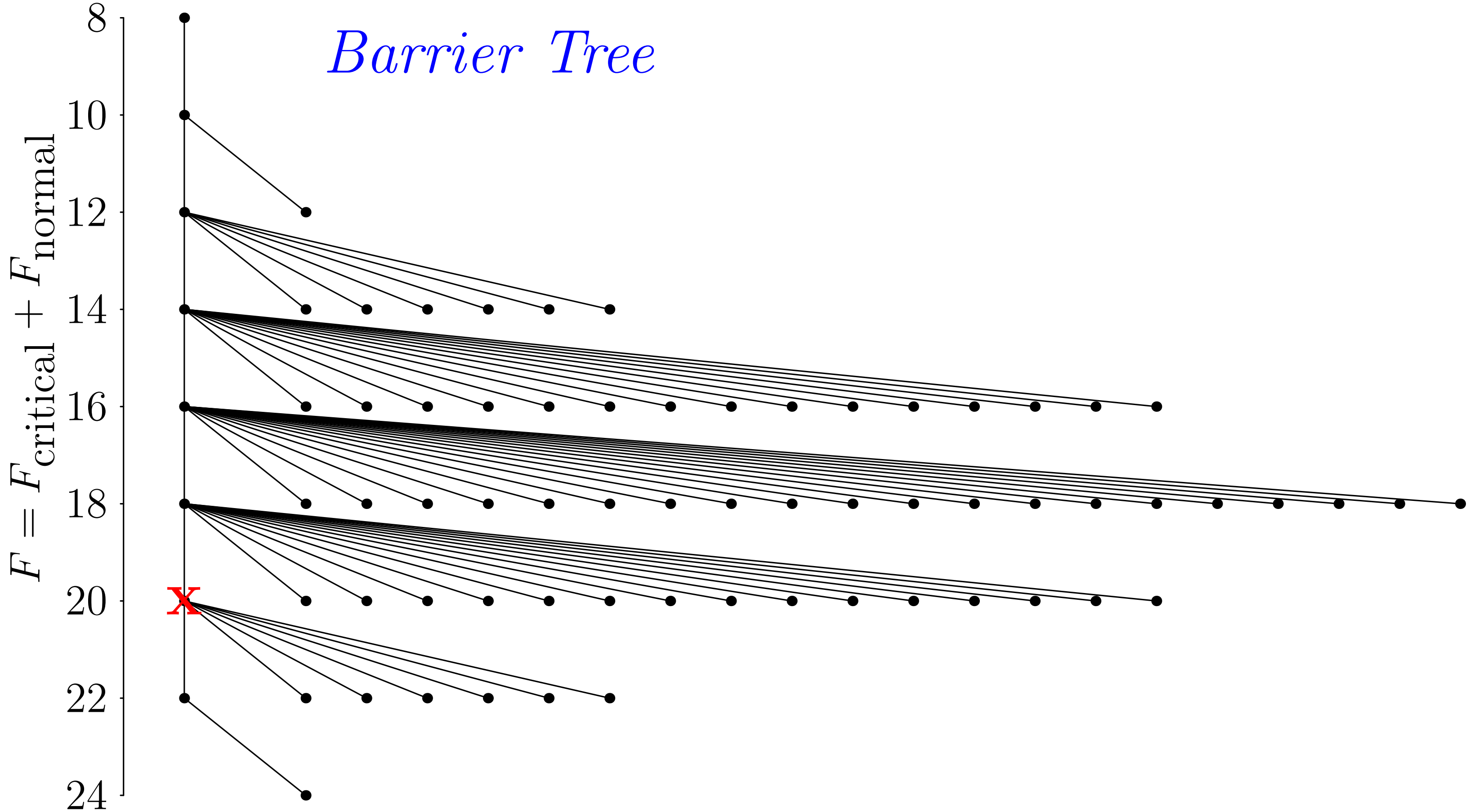
																								F
1	1	1	1	-1	1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	20



Fitness Landscape

Binary String

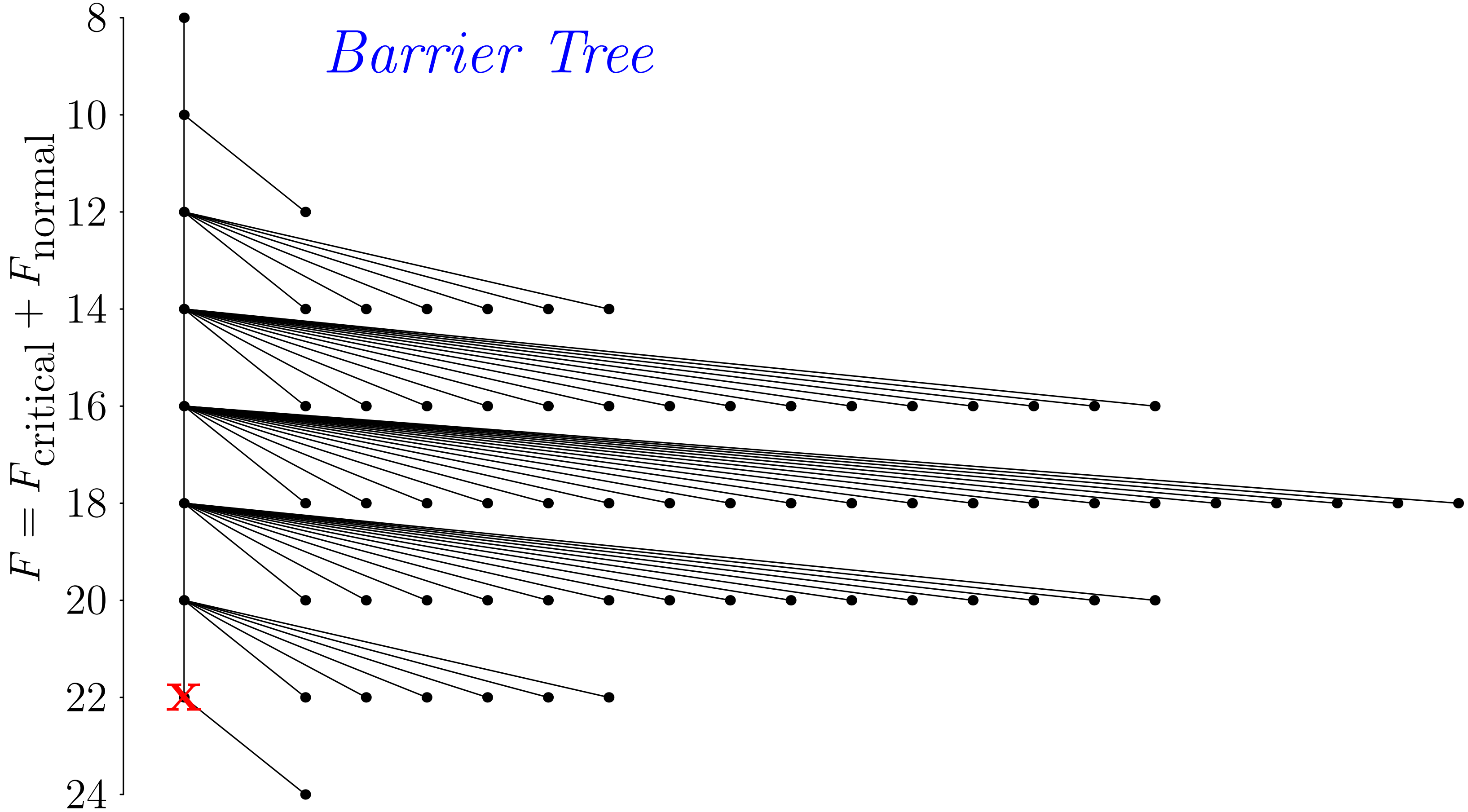
																								F
1	1	1	1	1	1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	20



Fitness Landscape

Binary String

																								F
1	1	1	1	1	1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	22



Fitness Landscape

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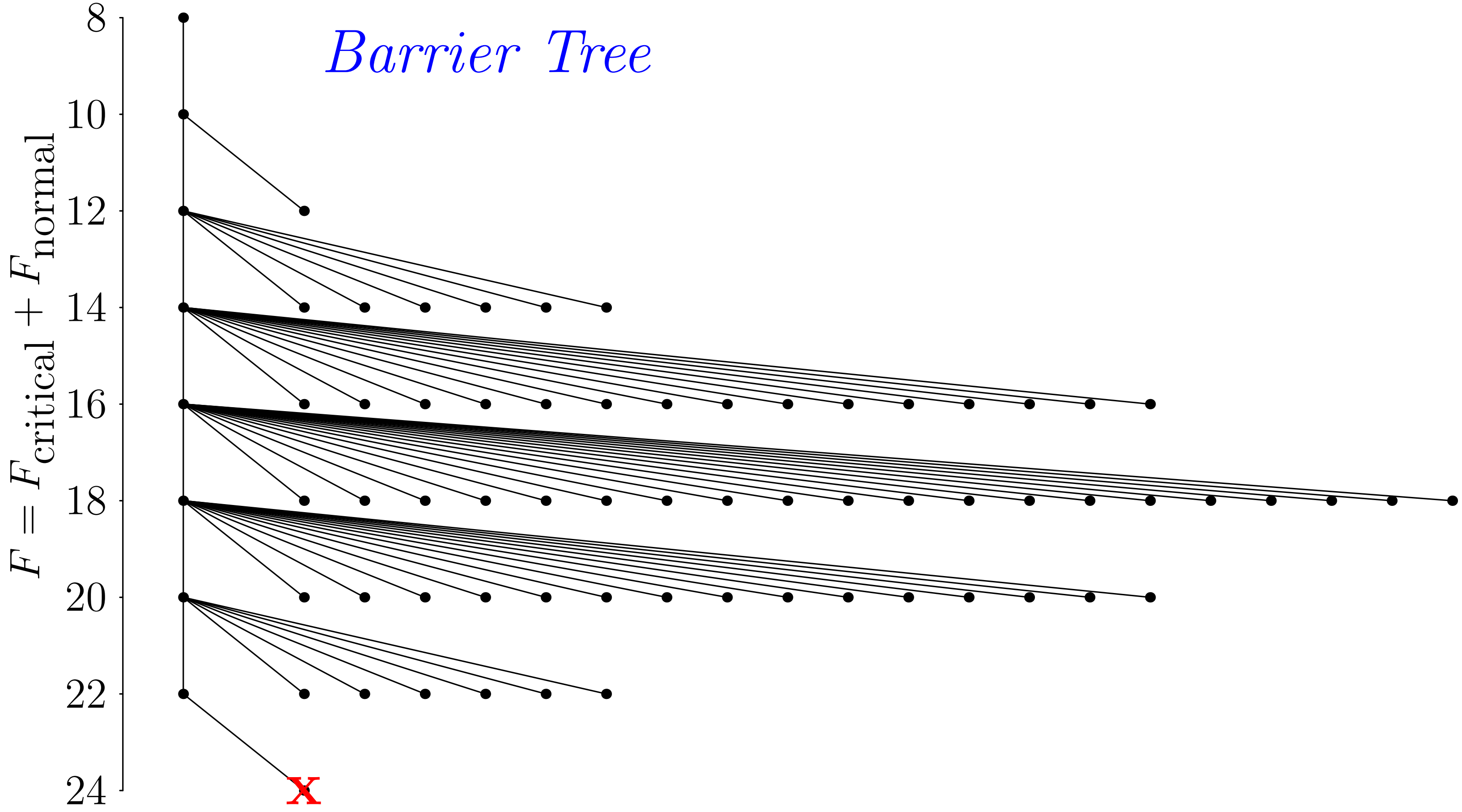
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	F 24



Simulation of GA

Initial Population

-1	-1	-1	-1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	1	-1	1	1	1	1	1	1	-1	0
-1	-1	1	1	-1	-1	-1	-1	-1	1	-1	-1	1	-1	1	-1	-1	1	-1	-1	1	1	1	1	4
1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	-1	-1	1	-1	1	-1	1	-1	1	1	1	-1	2
1	-1	-1	1	1	1	1	-1	-1	1	1	1	-1	1	-1	-1	-1	1	-1	-1	1	-1	1	-1	-2
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1	-1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	-1	-4
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1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	-8

Simulation of GA

Population after 1 hill-climbing step

-1	-1	-1	-1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	1	-1	1	1	1	1	1	1	1	2
-1	-1	1	1	-1	-1	-1	-1	-1	1	-1	-1	1	-1	1	-1	-1	1	-1	-1	1	1	1	1	4
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1	-1	-1	1	1	1	1	-1	-1	1	1	1	-1	1	-1	-1	-1	1	-1	-1	1	-1	1	-1	-2
1	-1	-1	-1	1	1	1	-1	1	1	1	1	-1	1	1	1	1	1	1	1	-1	-1	1	1	2
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1	-1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	-4

Simulation of GA

Population after 50 hill-climbing steps

-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	20
1	-1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	16
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-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	18
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-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	16
-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	16

Simulation of GA

Population after selection

-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	22
1	1	1	1	1	1	-1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	18
1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	22
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1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	22
1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	22
1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	22

Simulation of GA

Population after uniform crossover

-1	1	1	-1	1	1	1	1	1	1	1	1	1	-1	-1	1	1	1	1	1	-1	1	1	-1	8
1	-1	-1	1	1	1	1	1	1	1	1	1	-1	1	1	-1	1	1	1	1	1	-1	-1	1	10
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1	1	1	1	1	1	-1	1	1	-1	-1	-1	1	1	-1	-1	1	1	1	1	1	-1	-1	-1	6
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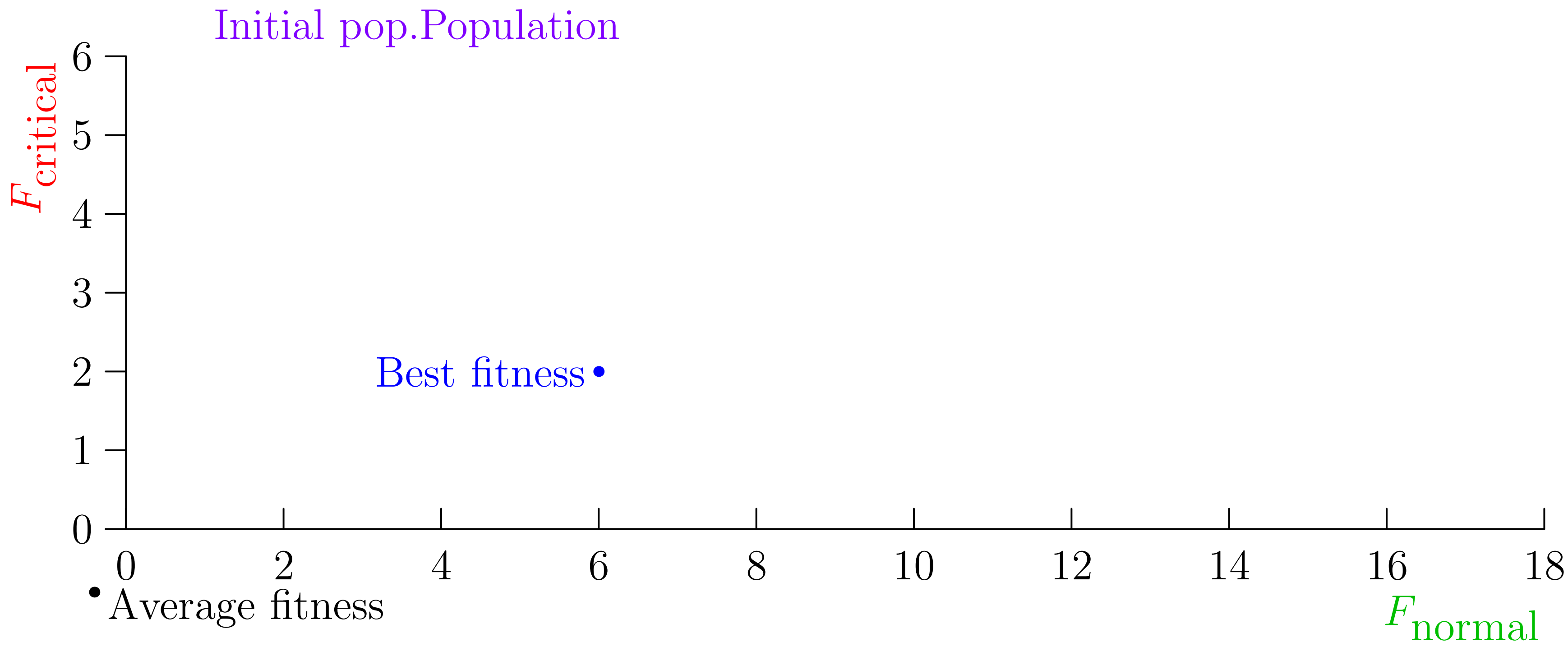
-1	-1	1	-1	1	1	1	1	1	1	1	1	1	-1	1	-1	-1	1	1	-1	1	1	-1	-1	6

Simulation of GA

Population after 50 hill-climbing steps

1	1	1	-1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	20
1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	22
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	22
1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	22
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	24
1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	22
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	24
1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	1	1	1	1	1	1	1	1	1	22
1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	22
-1	-1	1	-1	1	1	1	1	1	1	1	1	1	-1	1	1	1	1	1	1	1	1	1	1	18

Evolution of Fitness



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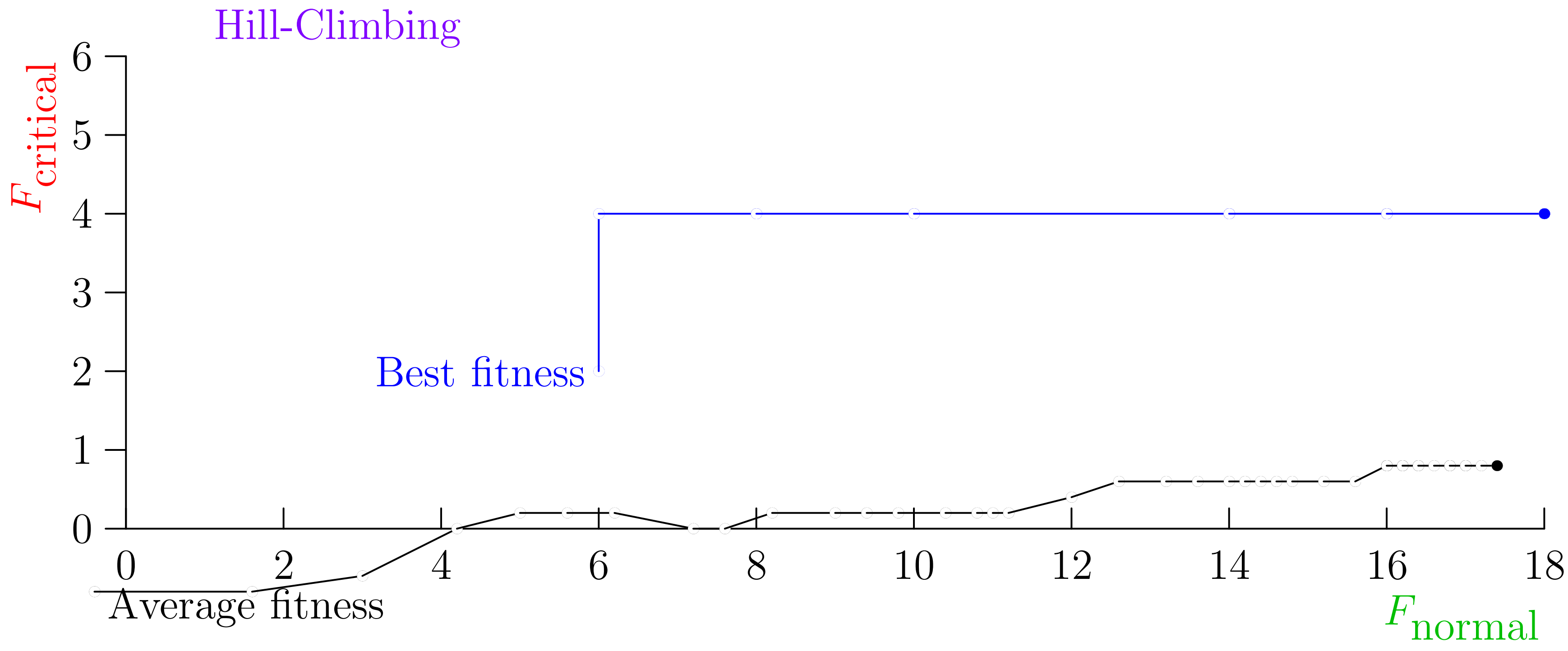

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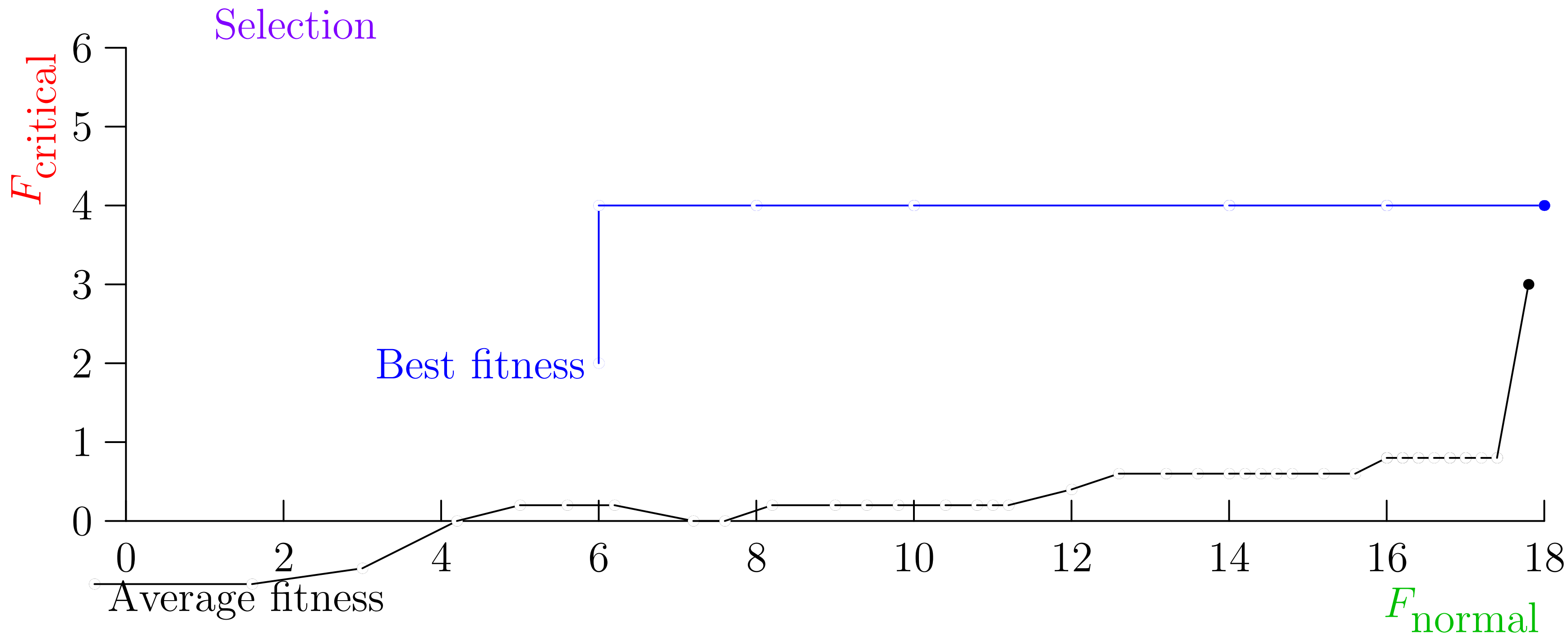

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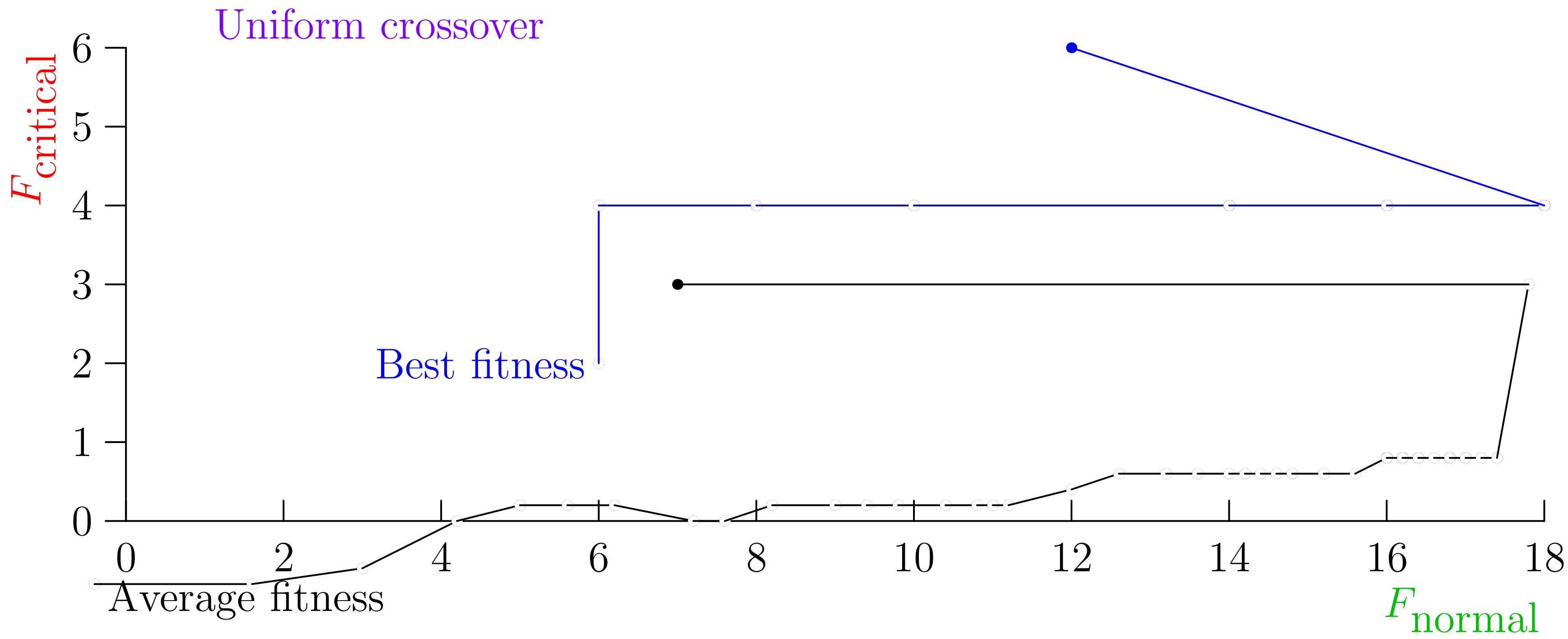

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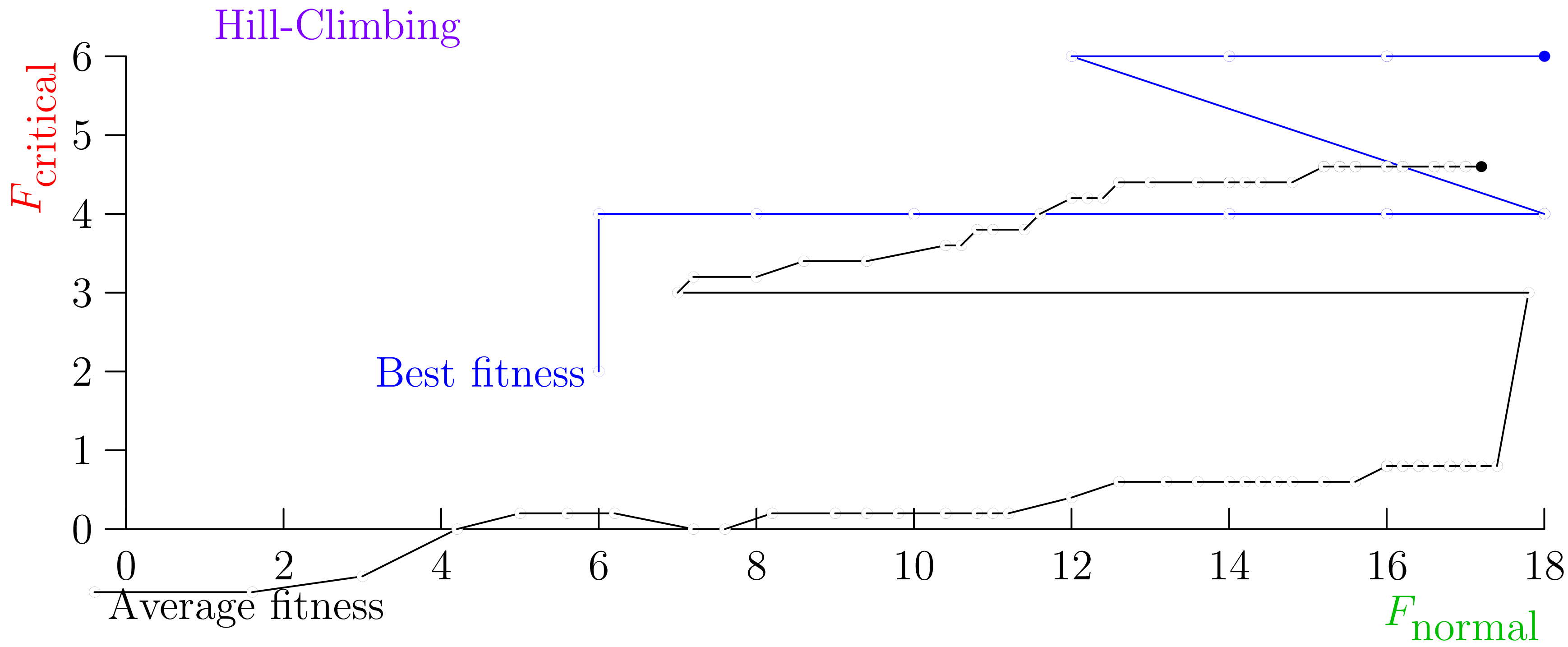
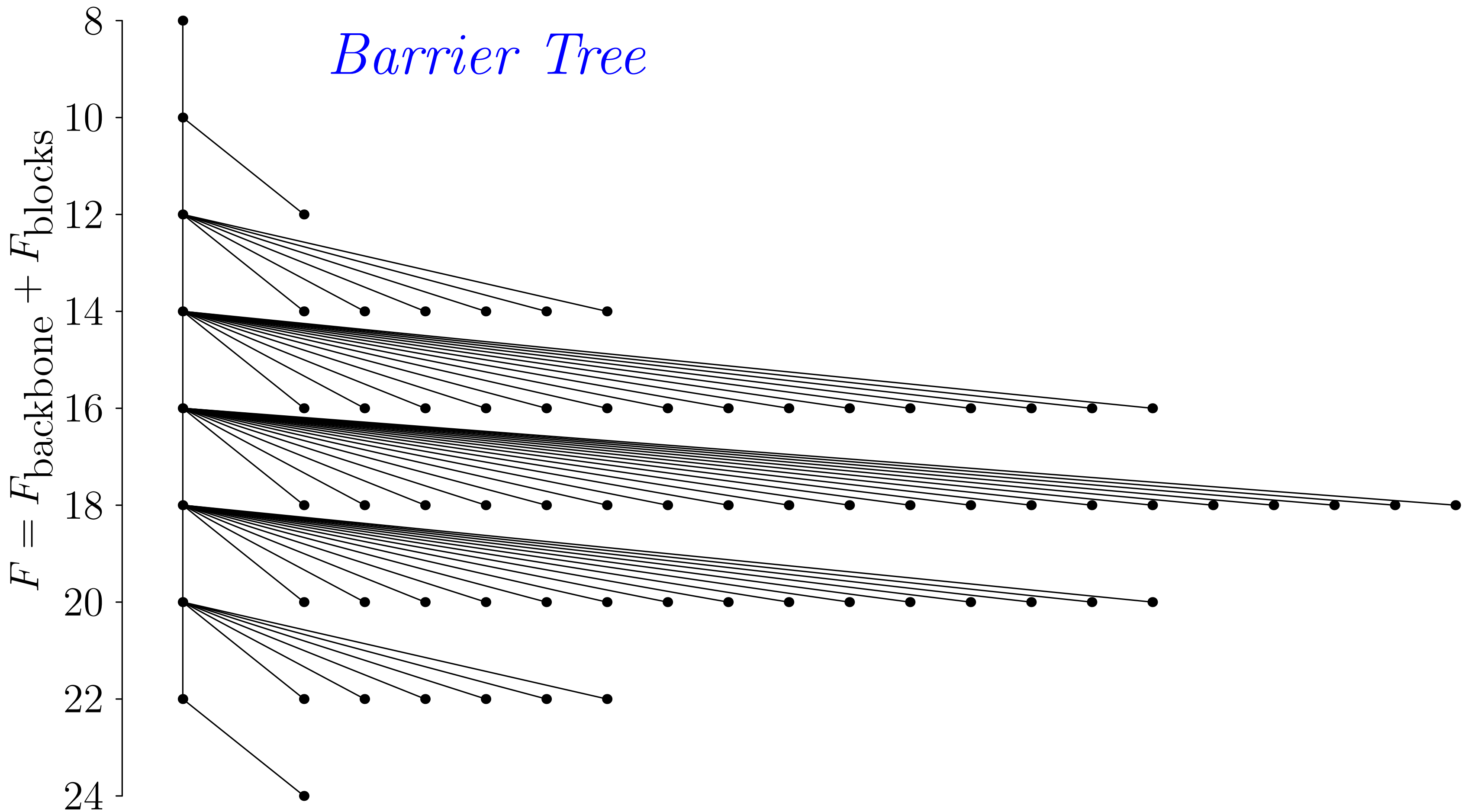


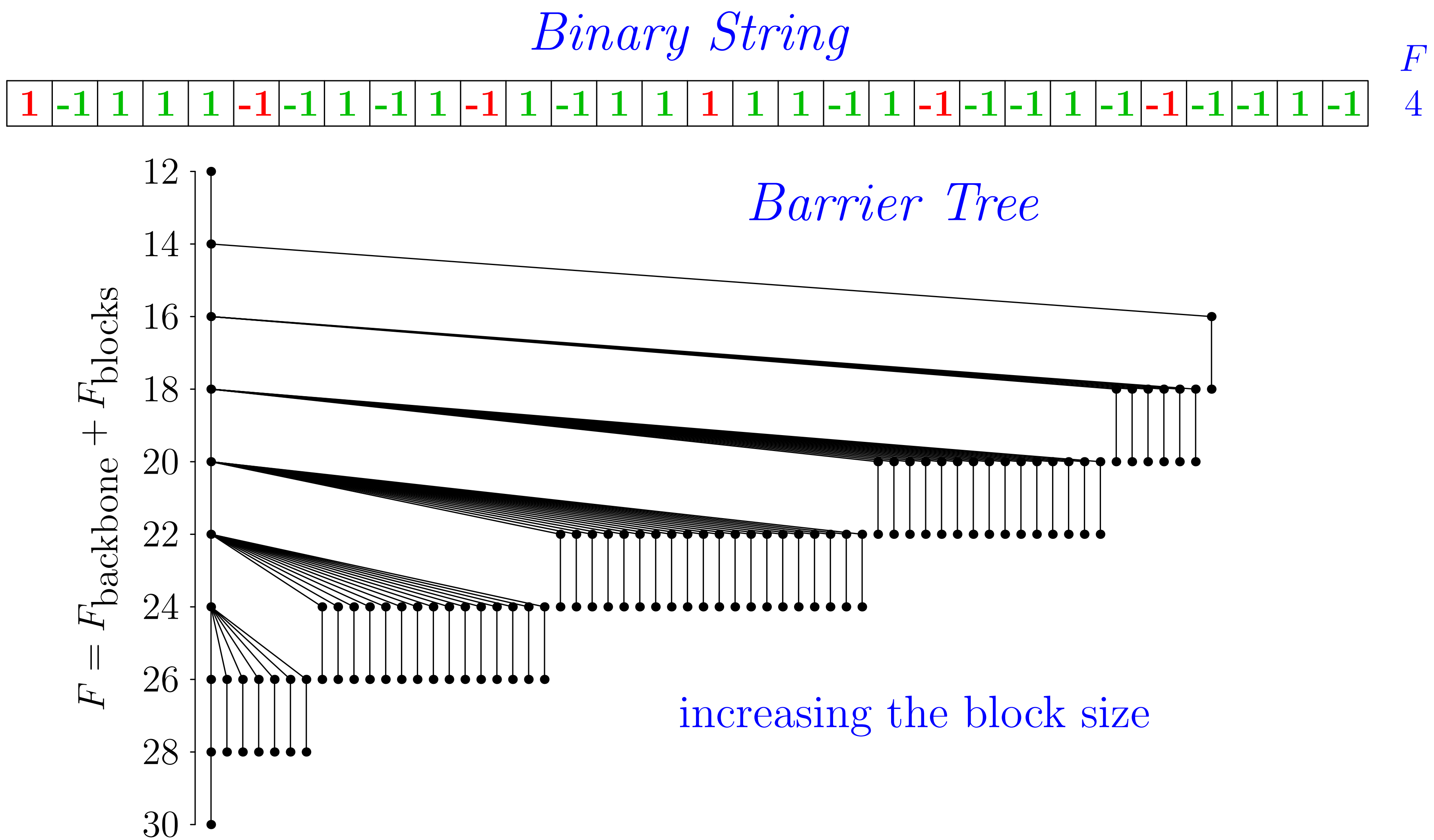
Making the Problem Harder

Binary String

																									F
1	-1	1	1	1	-1	-1	1	-1	1	-1	1	-1	1	1	1	1	1	-1	1	-1	-1	-1	1		-2



Making the Problem Harder



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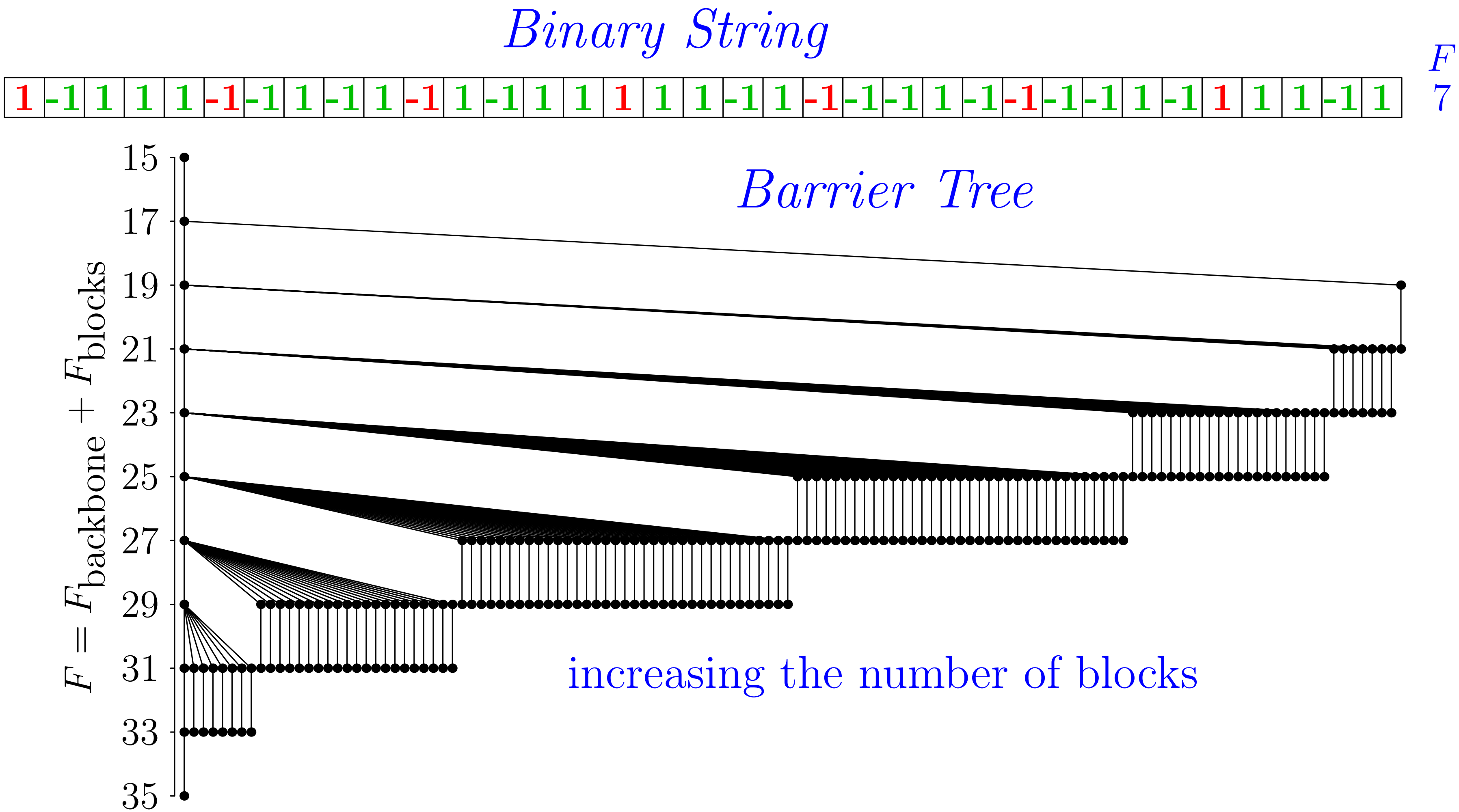

Provably Efficient?

- A challenge for algorithm complexity people

  Can you set this up so that a hybrid GA is provably polynomial while a hill-climber is exponential?

- Hybrid GA appears to be equivalent to solving ones-max using uniform crossover

- Difficulty is to show that fixation through hitchhiking won't occur

- However, for moderate selection strength, uniform crossover should decouple variables faster than selection replicates them

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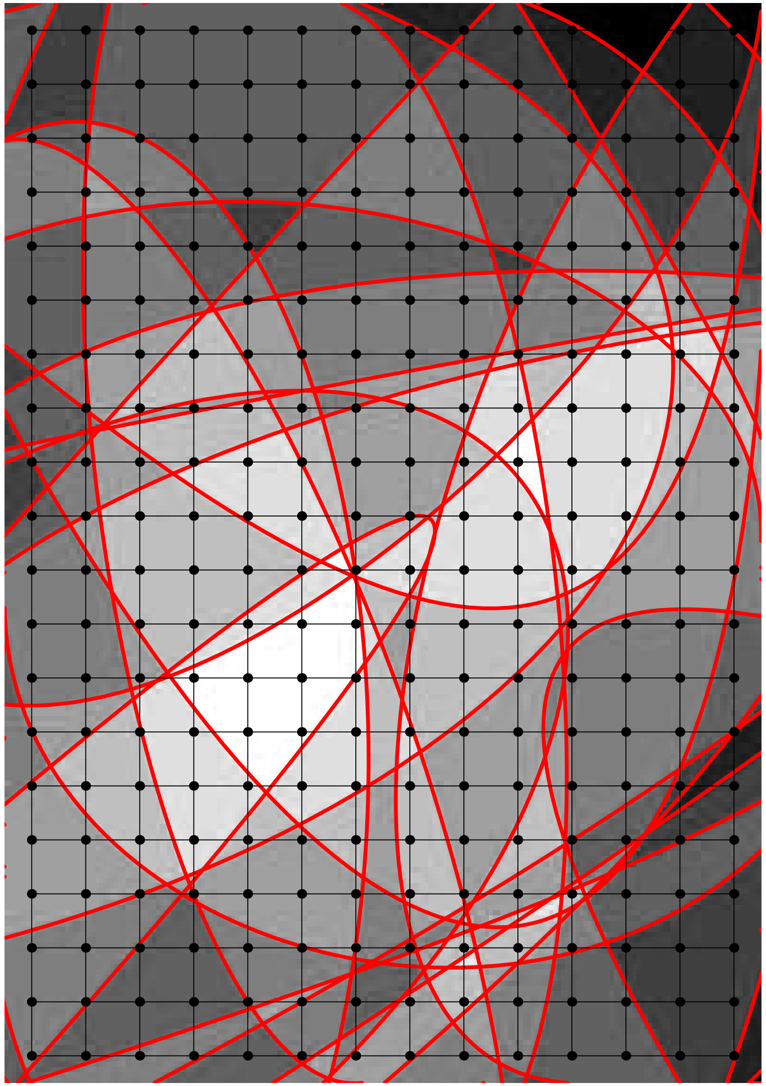
Outline

1. Critical Variable Problems
2. Solving Critical Variable Problems
3. A Toy Example
4. **Real Optimisation Problems**
5. Max-Sat



Common Observation

- A very common observation is that, for many hard problems, it can take many hours to find a good solution
- If you perturb the solution (e.g. by changing 10% of the variables) you get to a solution with a very poor fitness value
- But, if you then run a hill-climber you return to the same quality of solution in a fraction of a second even though initially it took hours to reach the solution
- This would be explained if the problem was a critical variable problem

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Backdoor

- Recently the idea of a backdoor has be put forward
- These exactly match on to the definition of critical variables

  They are a small set of variable which when set makes the problem easy
- They have been studied (empirically) in the context of very small SAT problems
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- How can this be?

Easy NP-Hard Problems

- Not all instances of NP-hard problems are difficult

- What is the shortest tour of this TSP problem?

- In fact, for some problems (e.g. subset-sum) almost all instances seem to be easy

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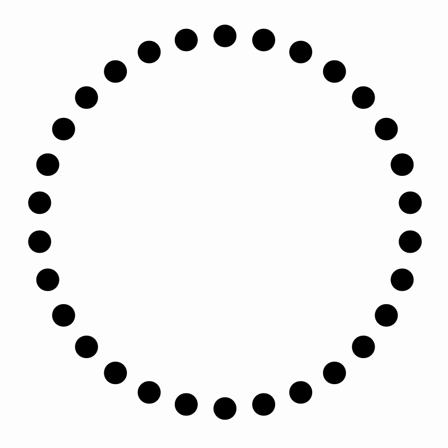

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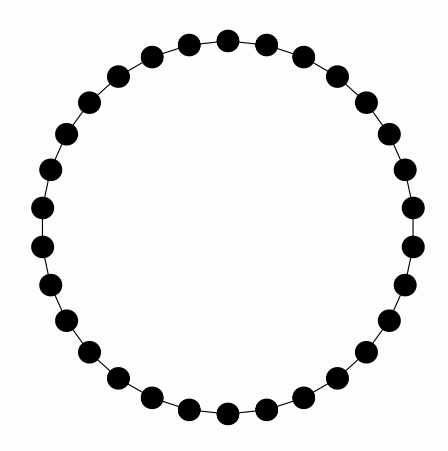

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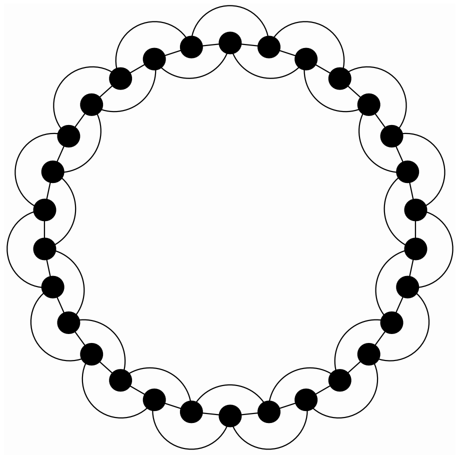

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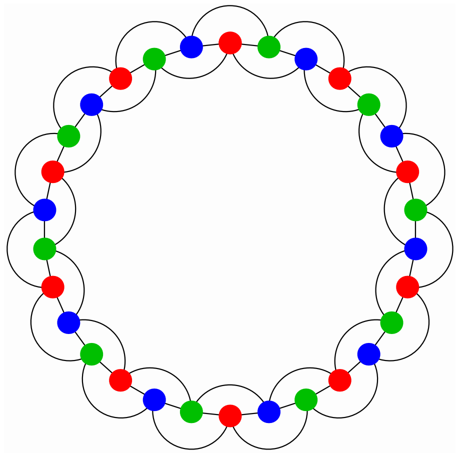

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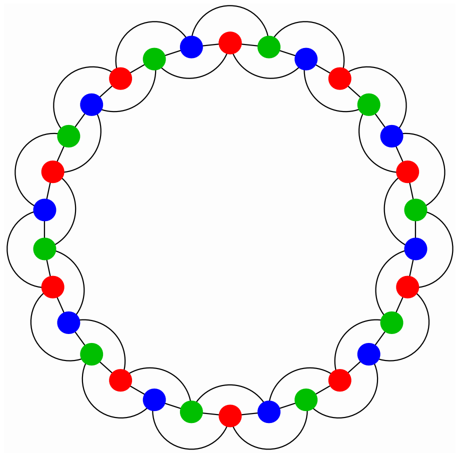

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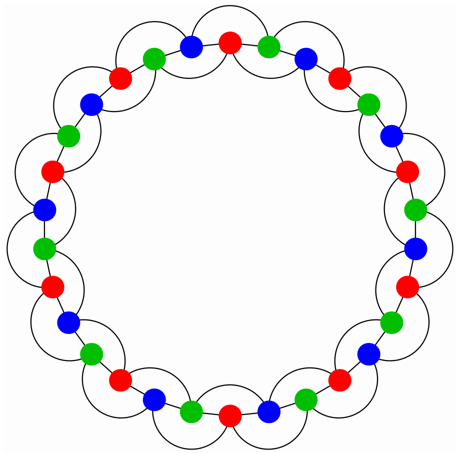

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Phase Transitions

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- When the number of constraints in the problem are small, almost all instances of the problem are easy to solve
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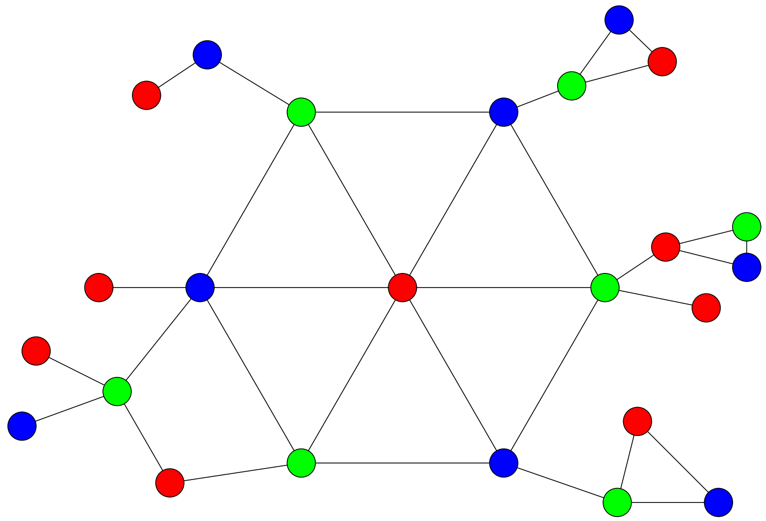
Example of Phase Transitions

- In graph-3-colouring, instances becomes hard as the probability of an edge reaches a critical value



- In SAT (finding an assignment of Boolean values that makes a formula written in CNF true), instances become hard when the ratio of clauses to variables reaches a critical value

$$(X_1 \vee \neg X_2 \vee X_3) \wedge (\neg X_1 \vee X_3 \vee X_4) \wedge (\neg X_2 \vee \neg X_3 \vee \neg X_4)$$

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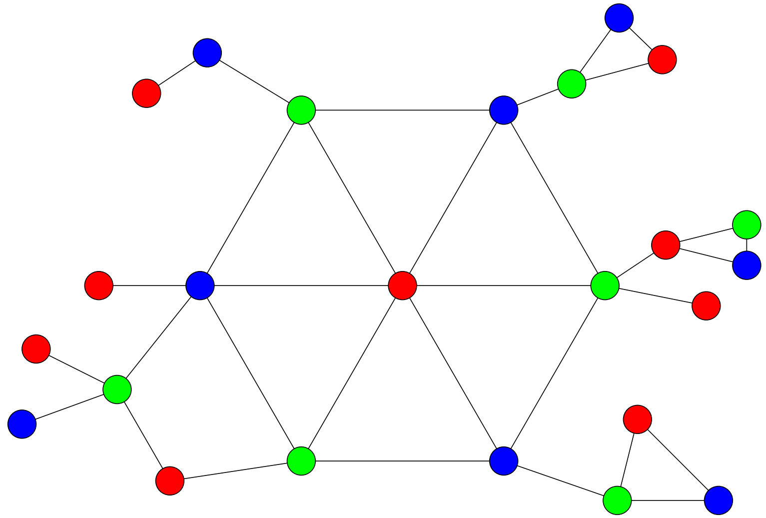

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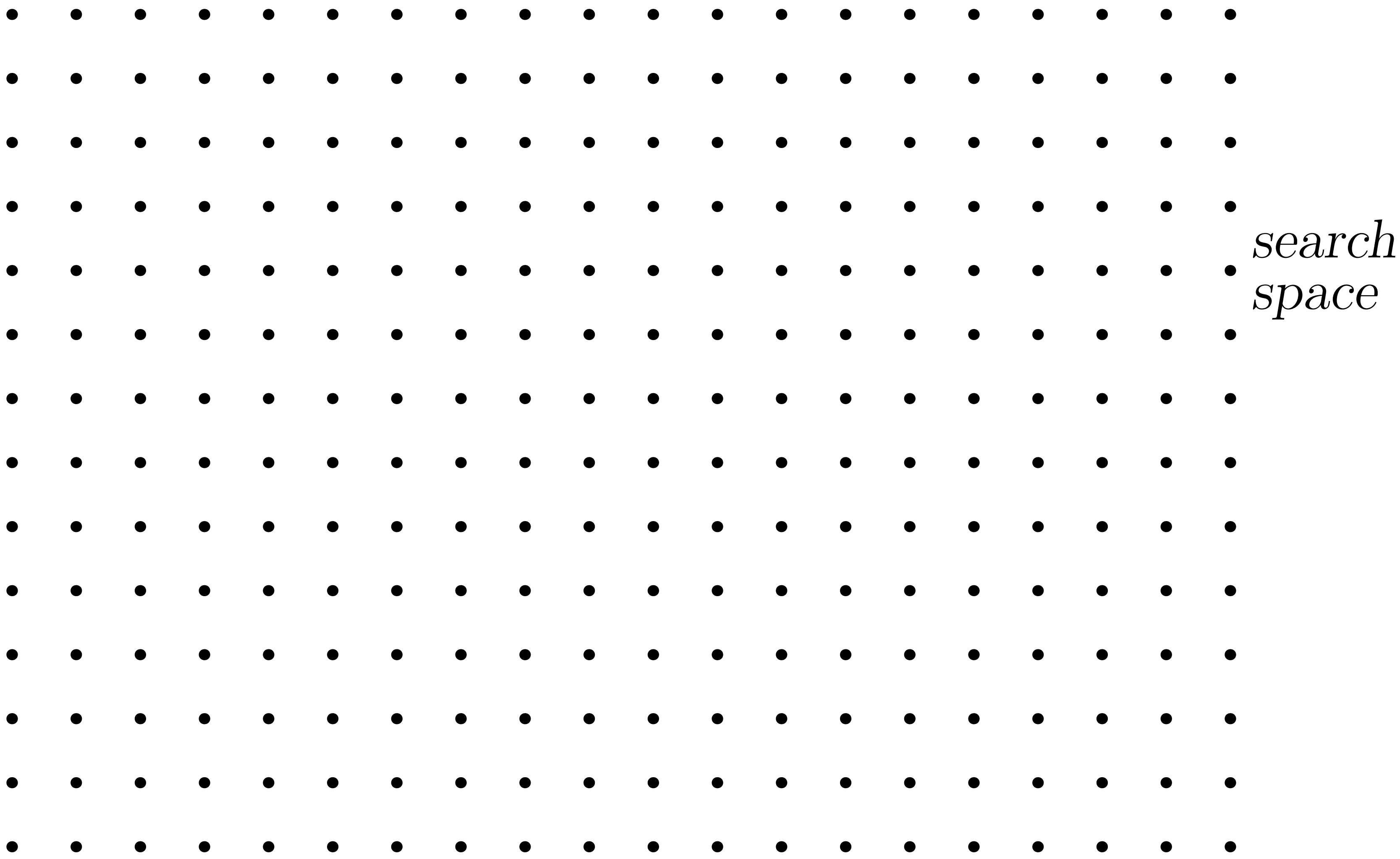
Schematic of Phase Transition



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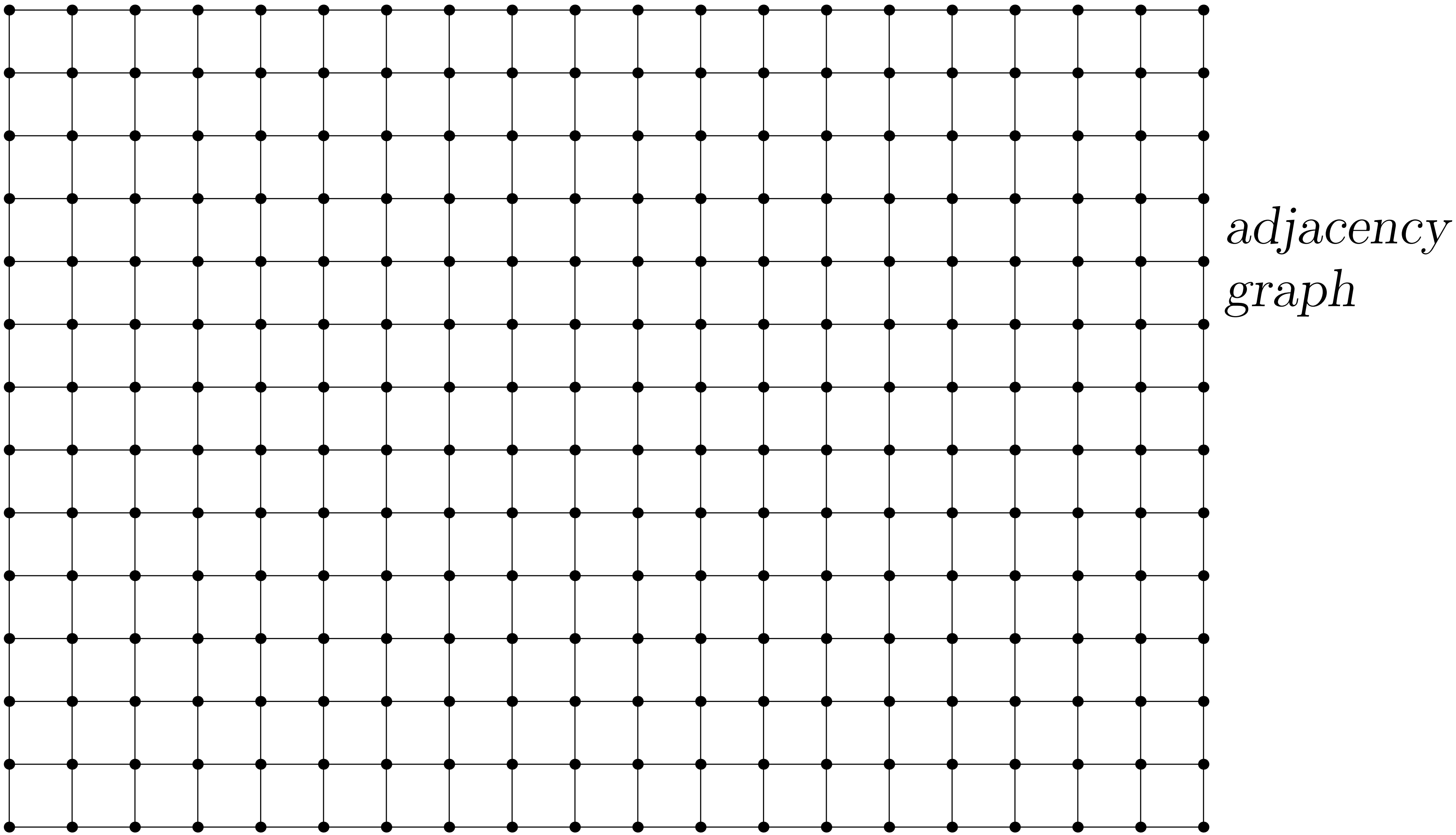

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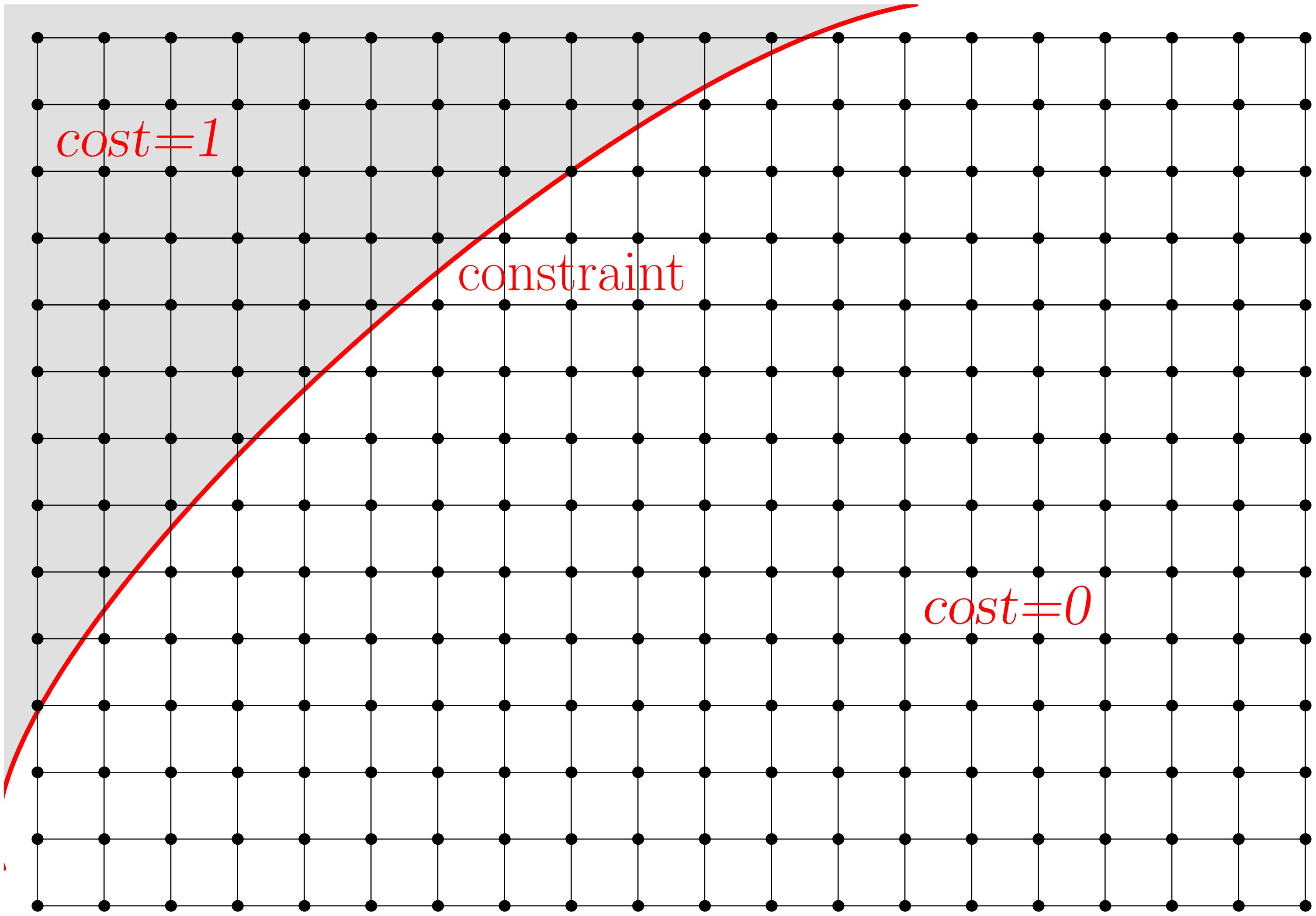

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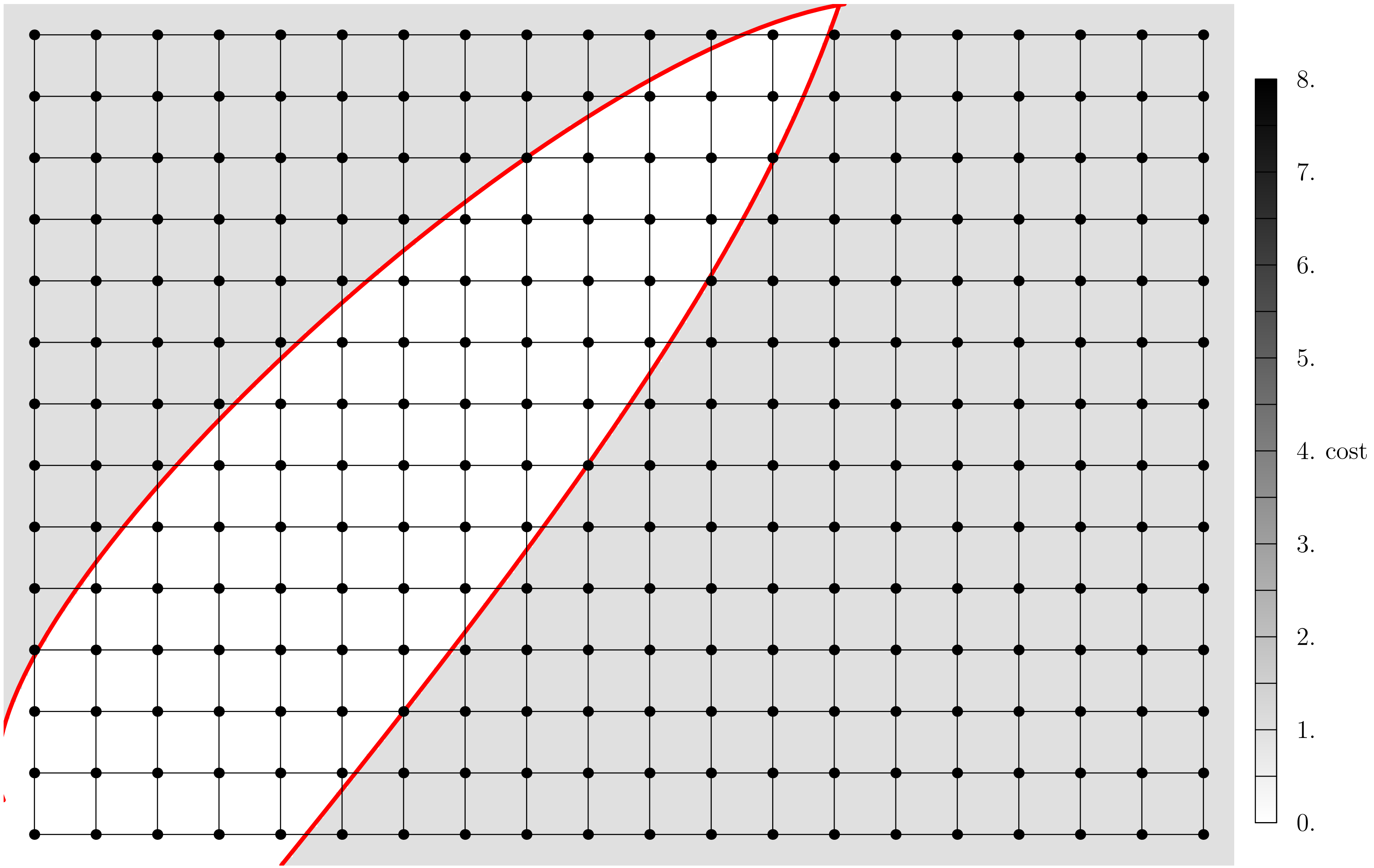

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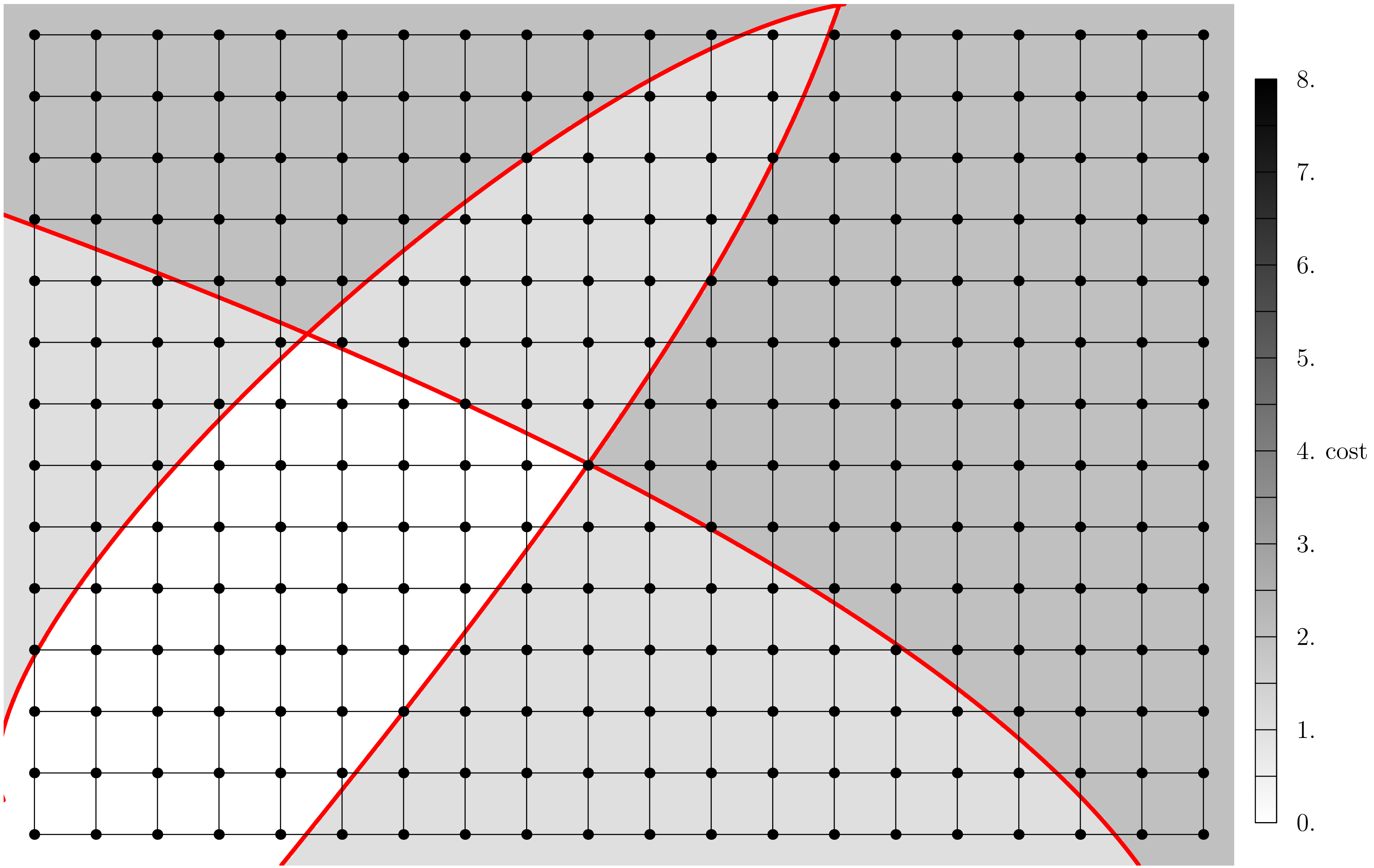

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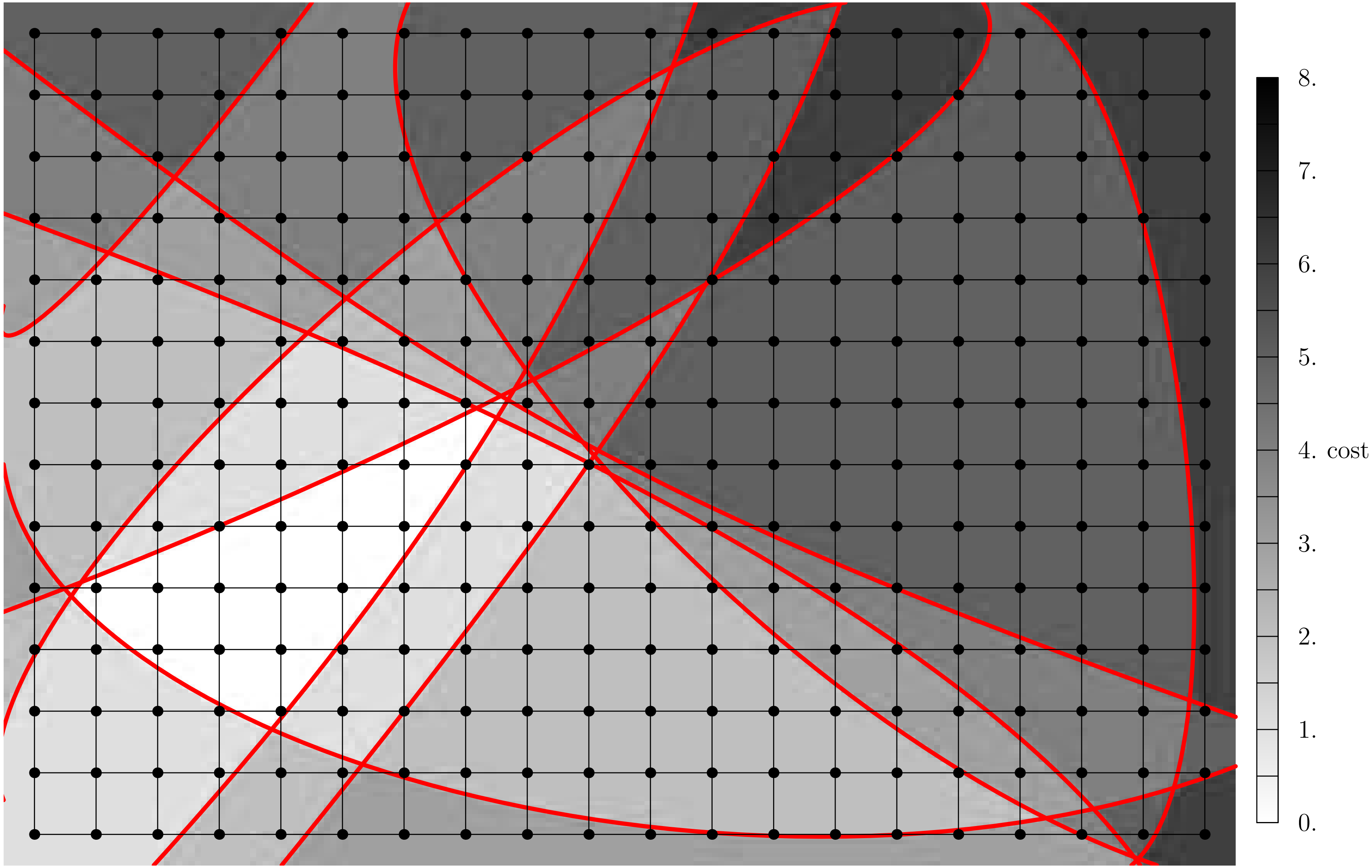

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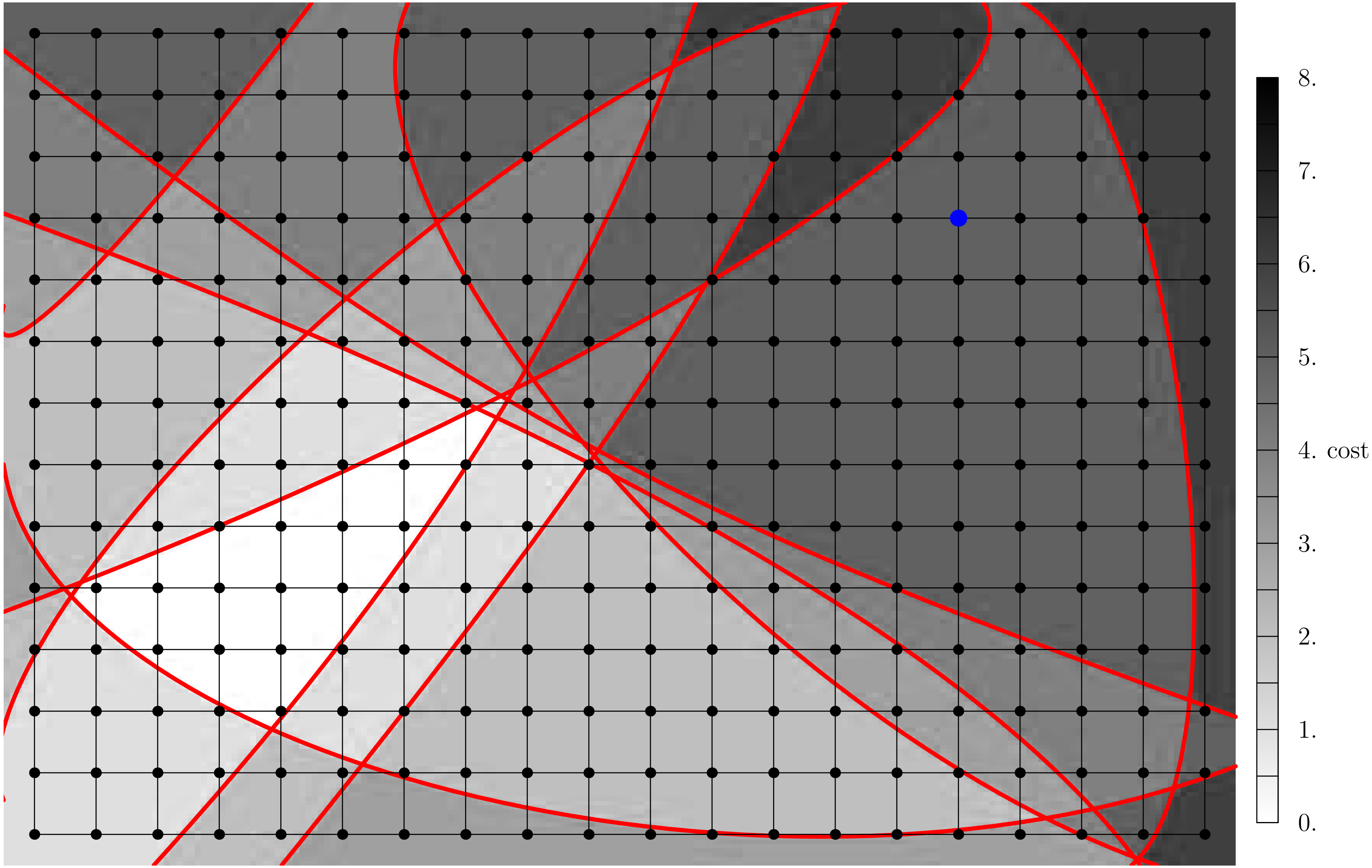

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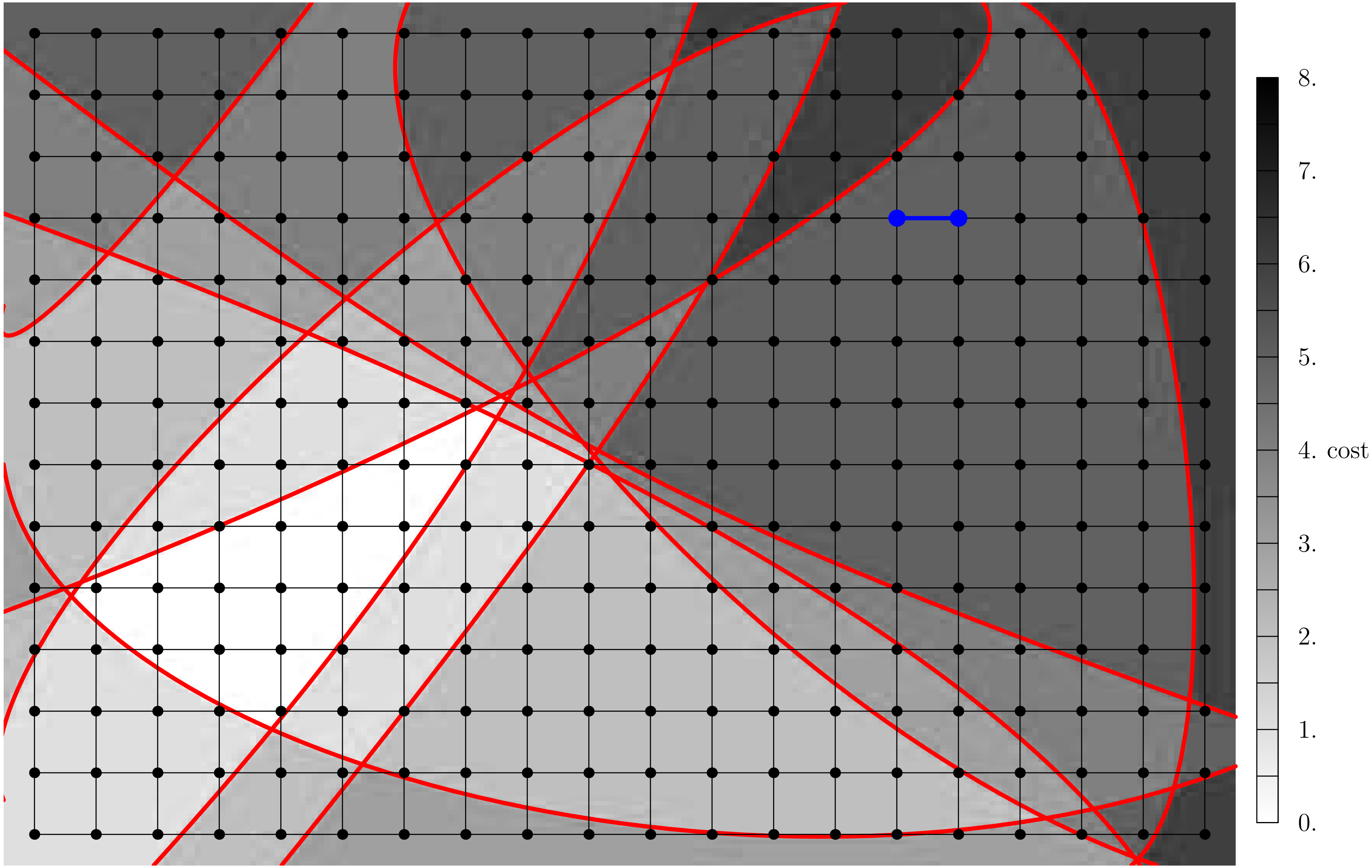

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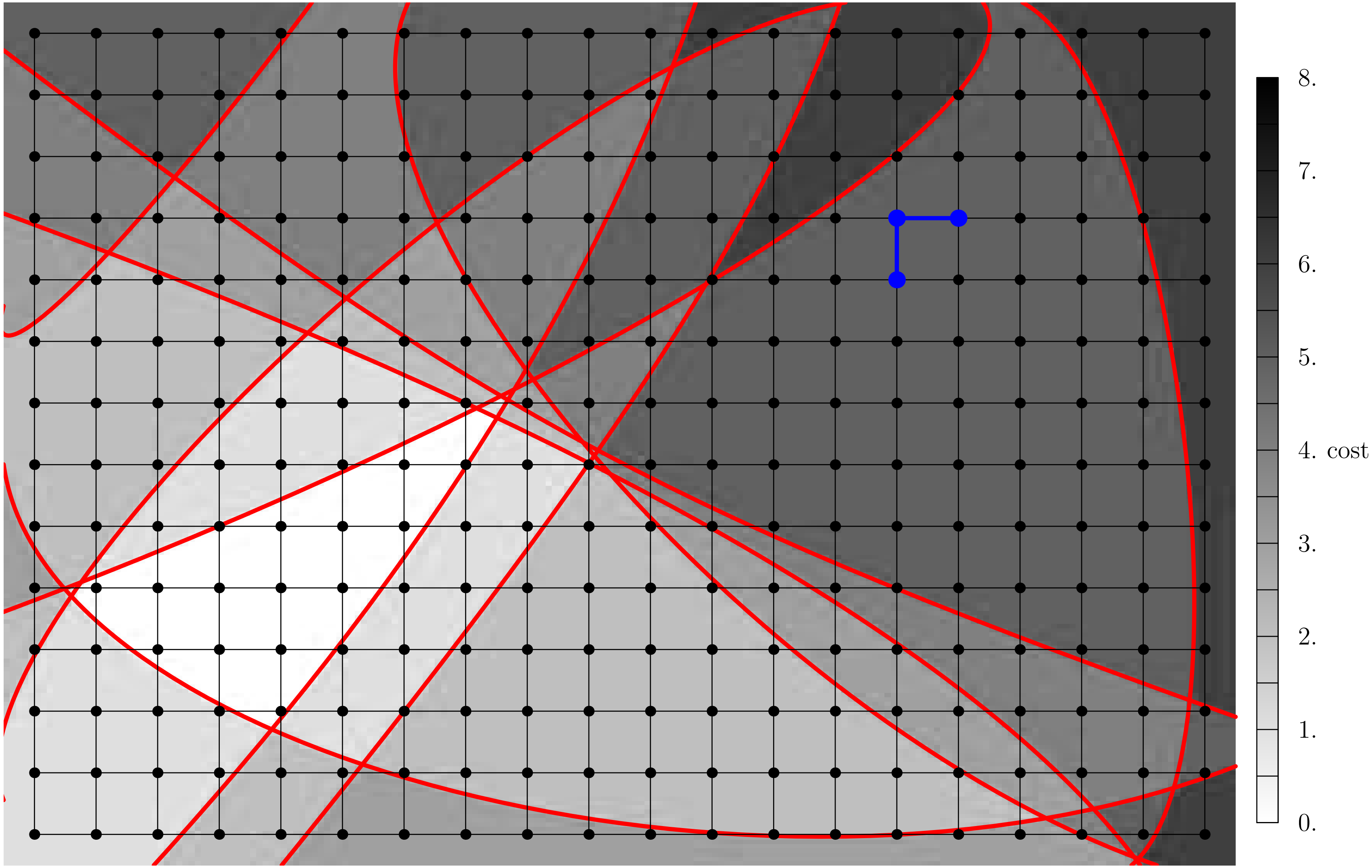

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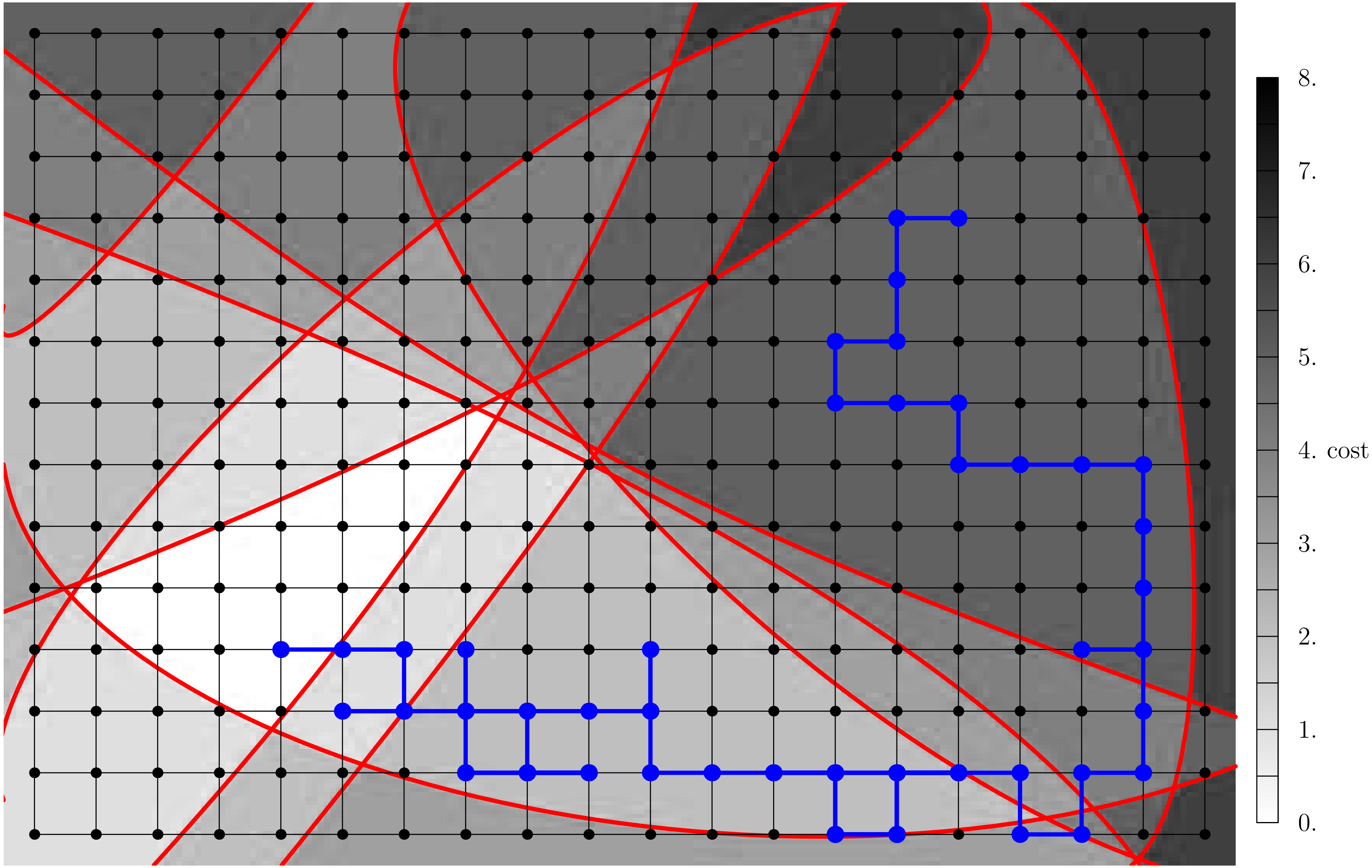

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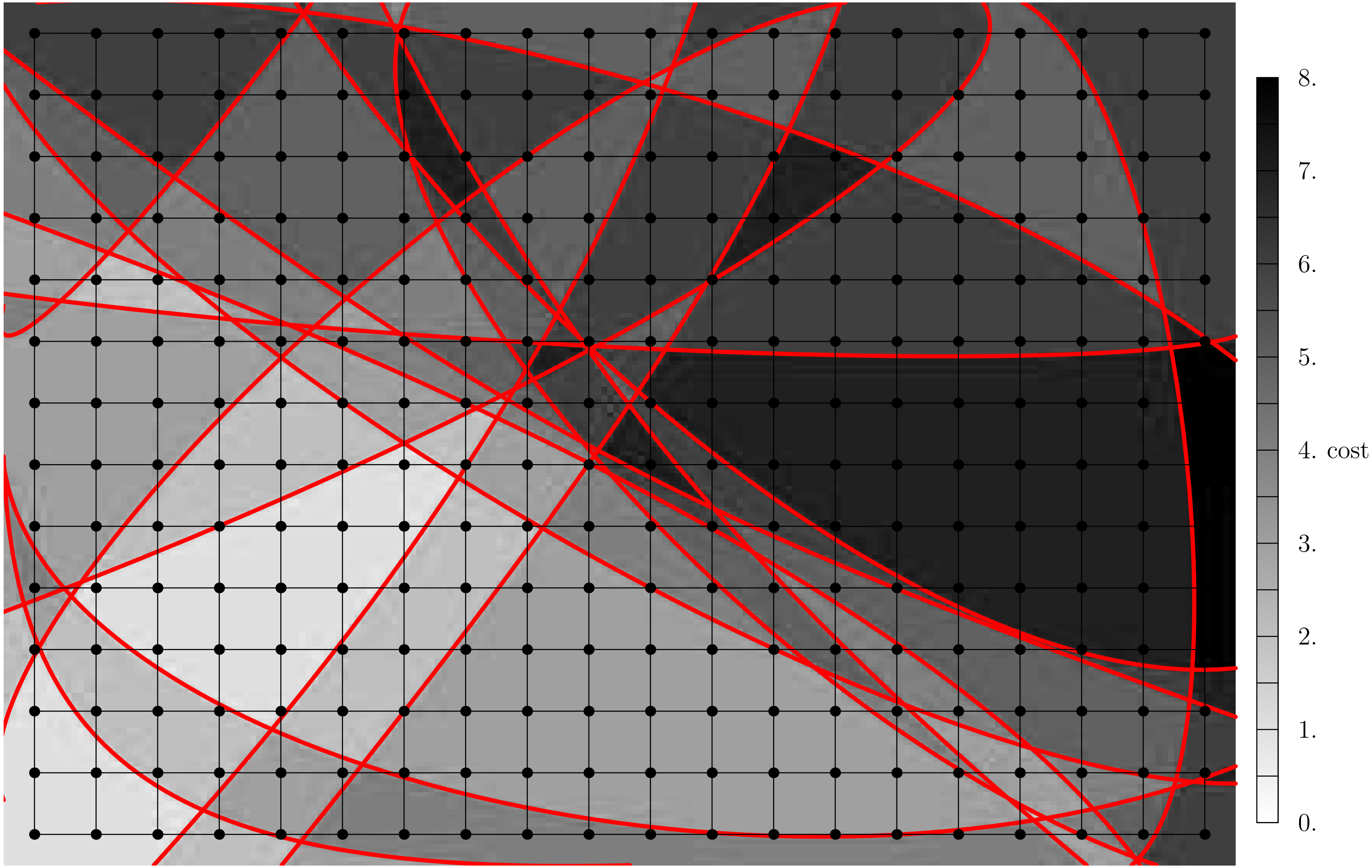

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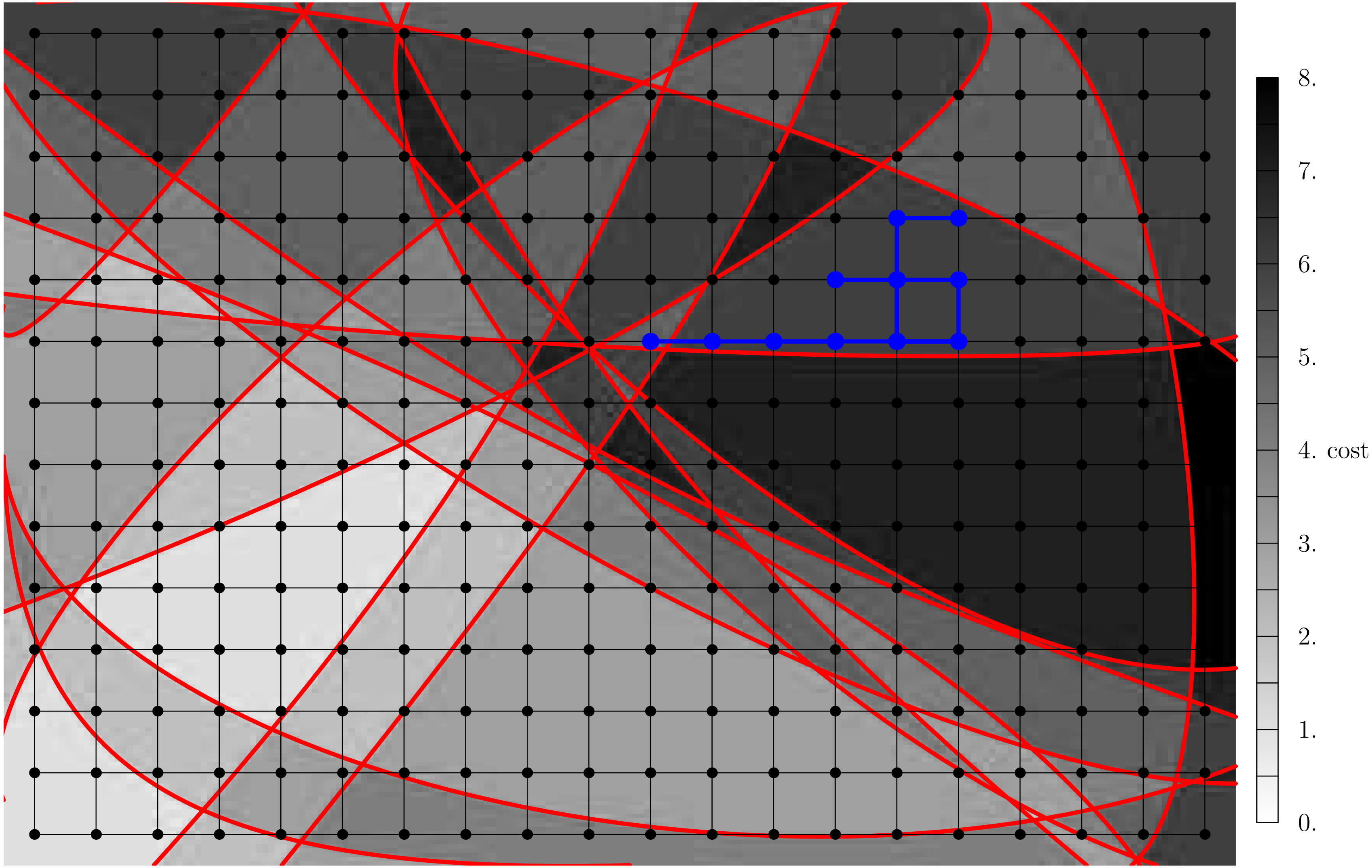
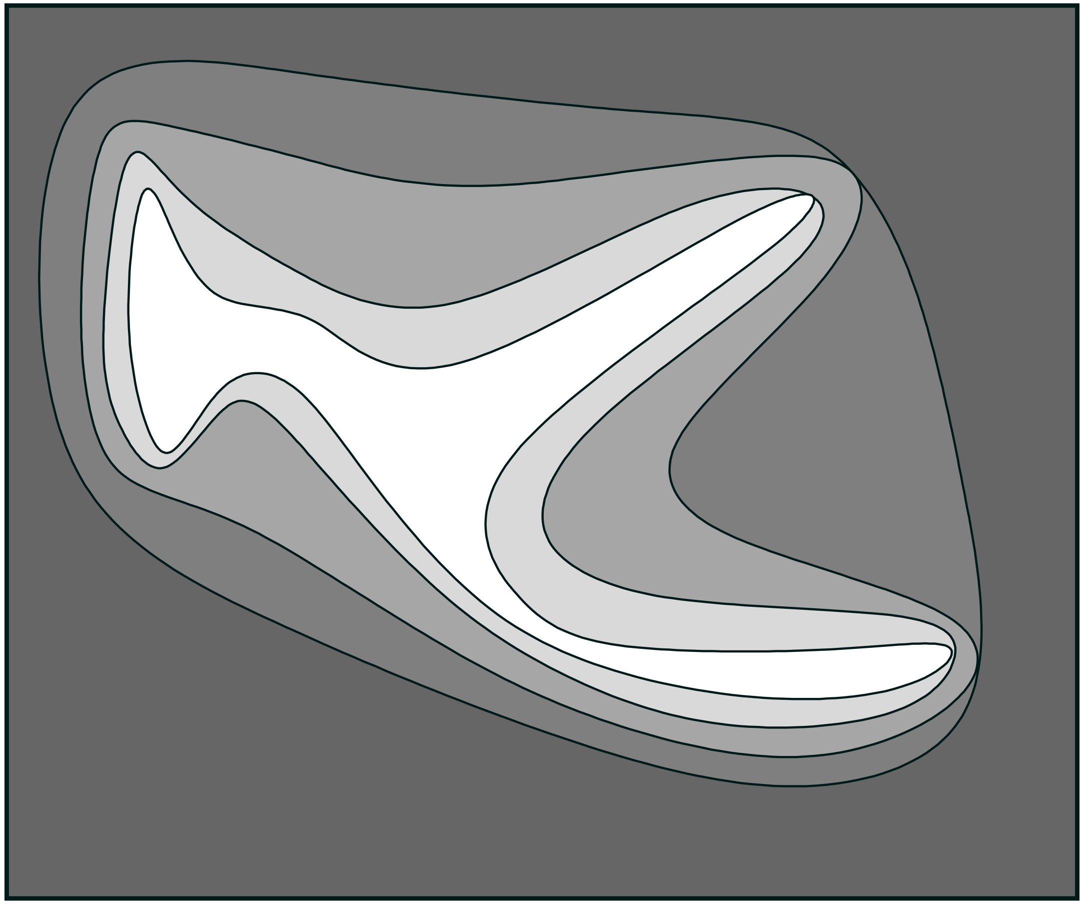


Schematic of Fitness Landscape

Global optima in white



Easy Phase $\alpha < \alpha_c$

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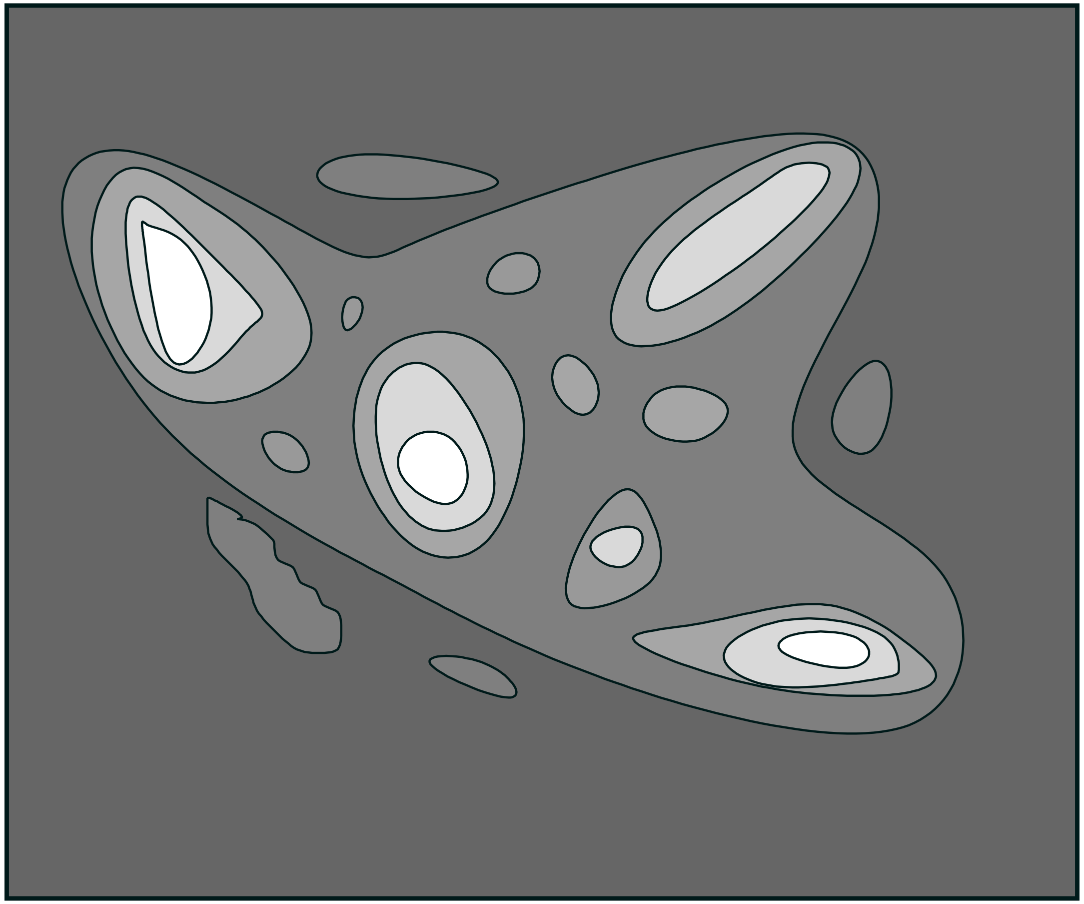
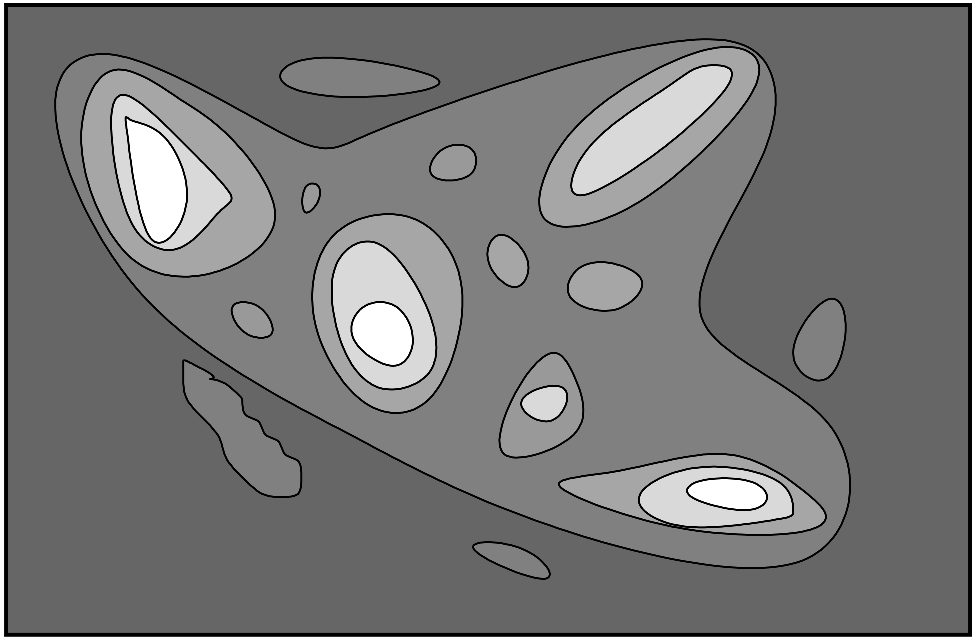


Hard Phase $\alpha > \alpha_c$

Critical Variables

- The set of critical variables can be thought of as

  a core of the solution which we have to get right to be in the correct part of solution space—pushes the problem back into the easy phase



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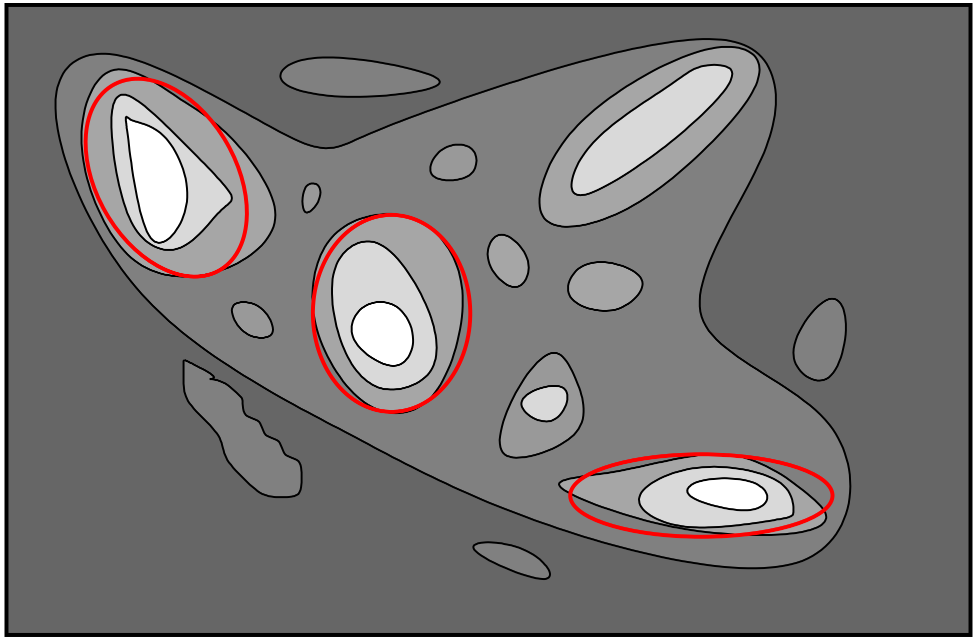

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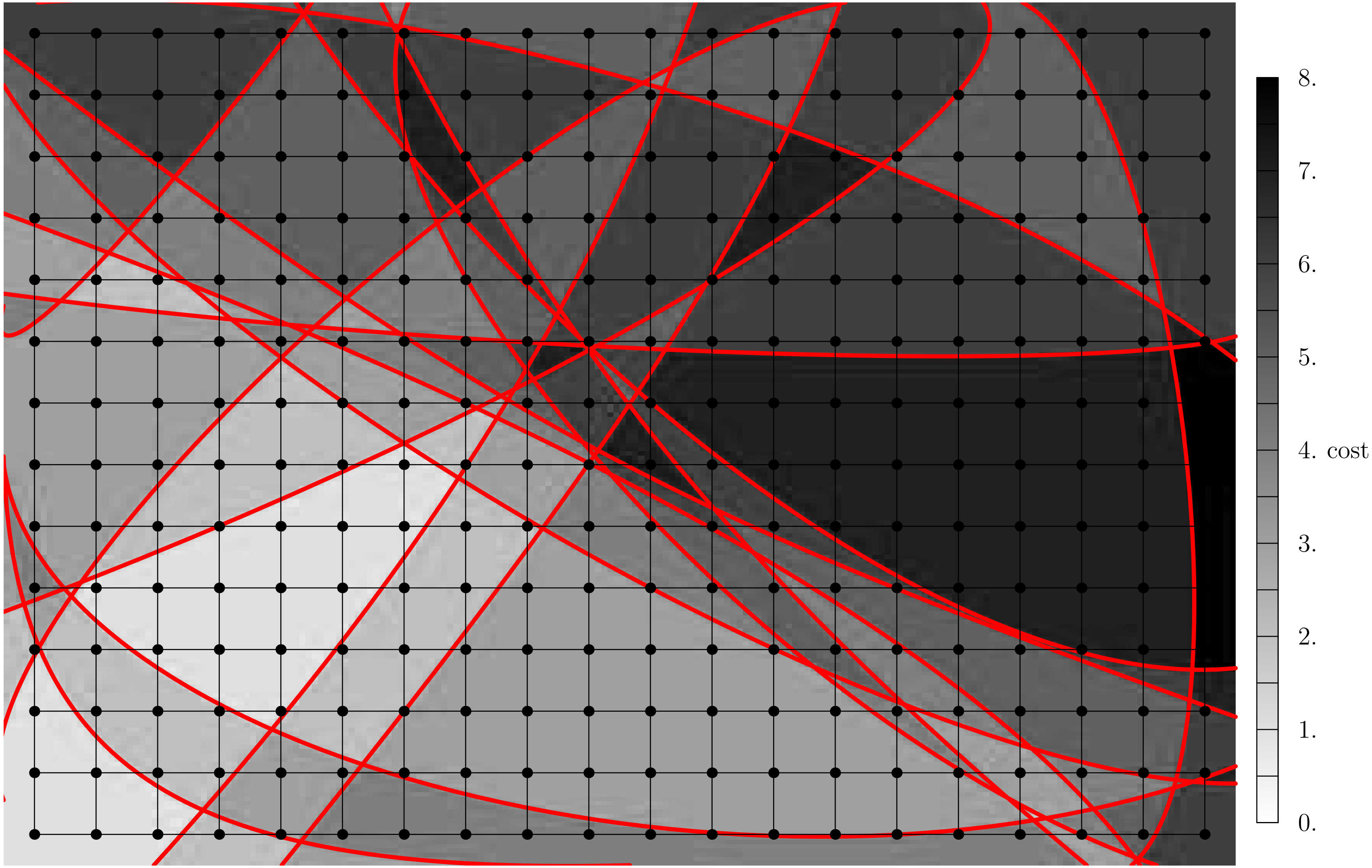

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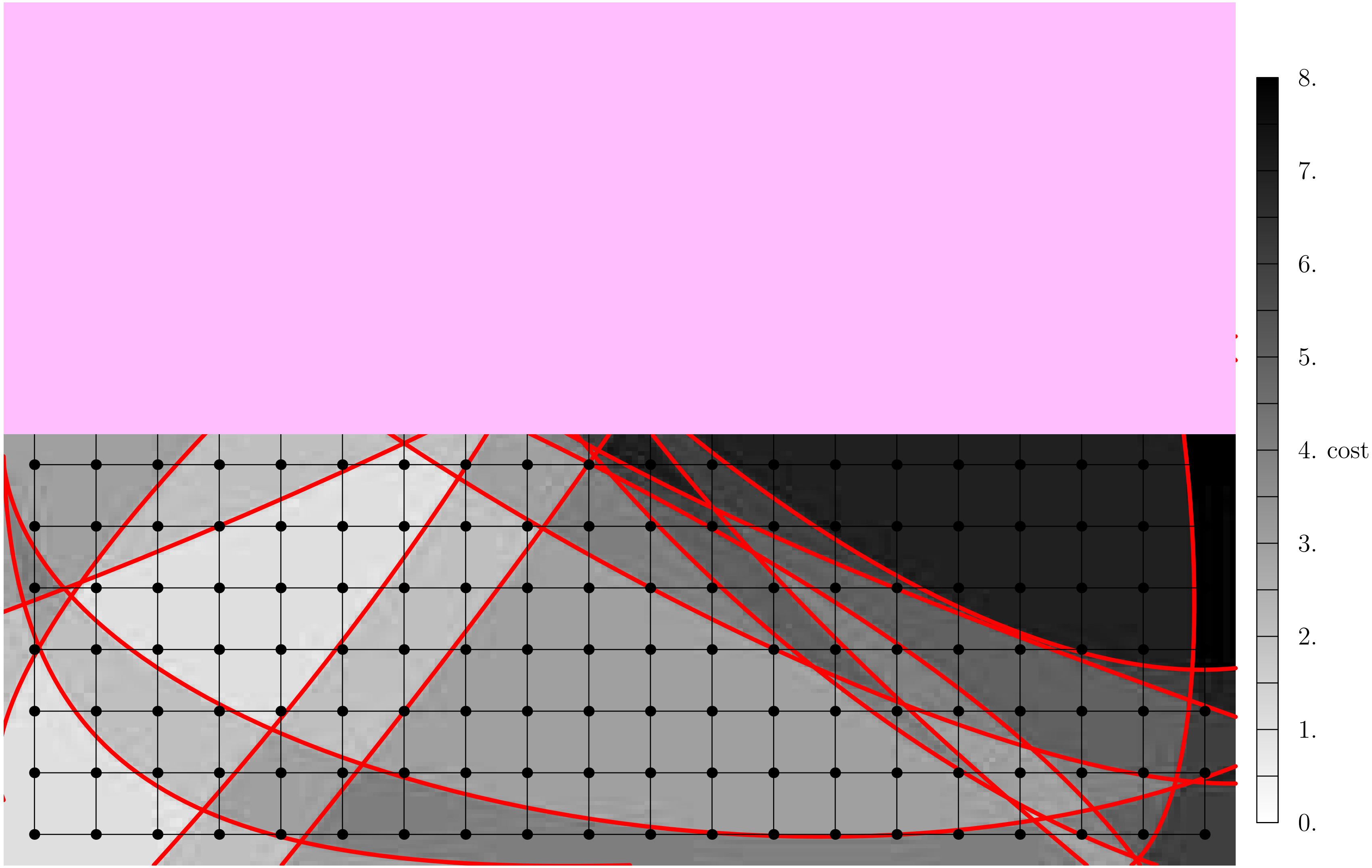

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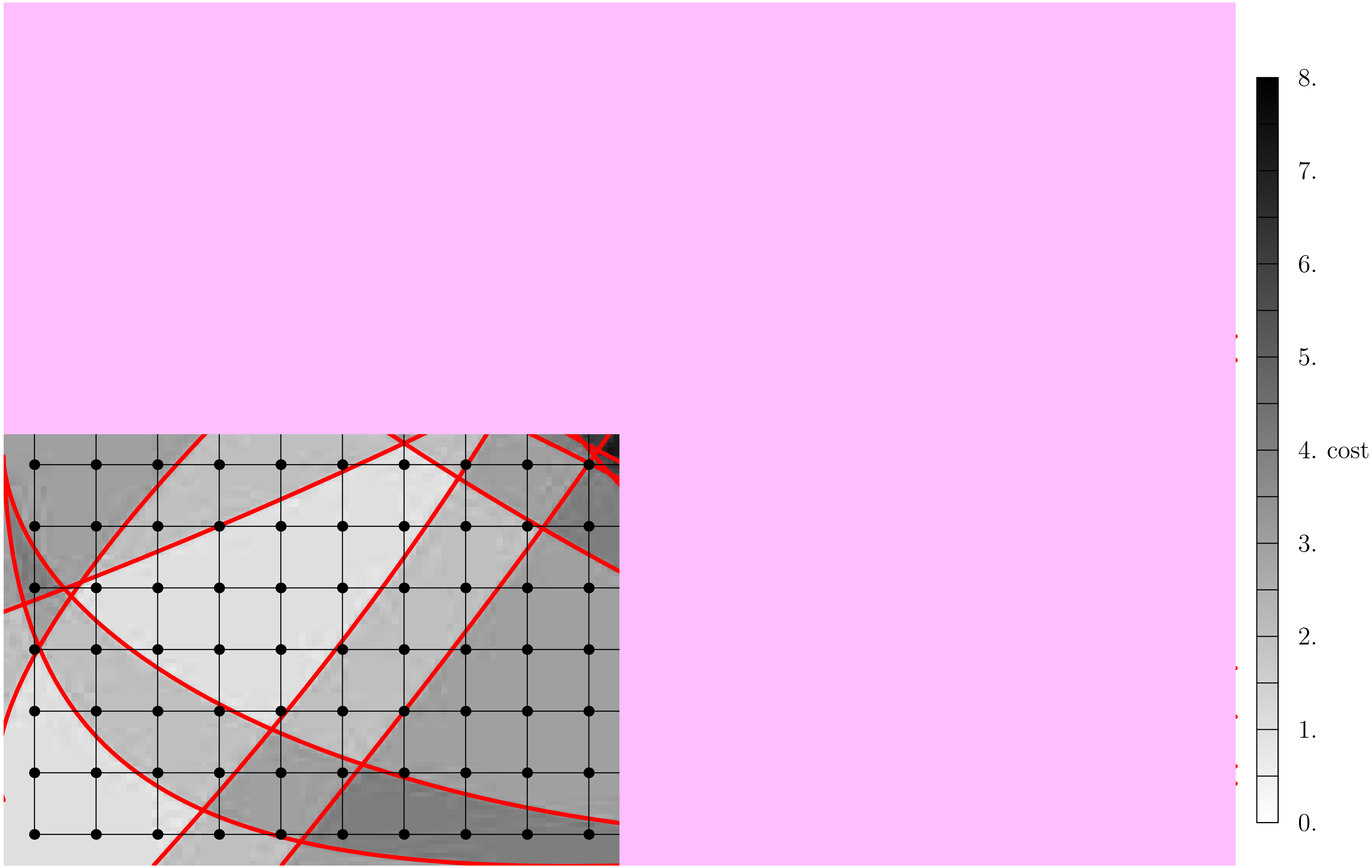

What Are Critical Variables?

- In TSP they would be critical edges
- Once in place a greedy algorithm would easily join the remaining cities
- In graph colouring they would be pairs of nodes that have to be coloured the same
- The rest of nodes would all fall into place
- In SAT they would a few variables that had to be set in a particular way
- All other variables would then be found using a hill-climber

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  - ⋆ Used a hybrid GA
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  - ⋆ And a clever crossover
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- The crossover operator seemed to act as an efficient method for finding good critical variables
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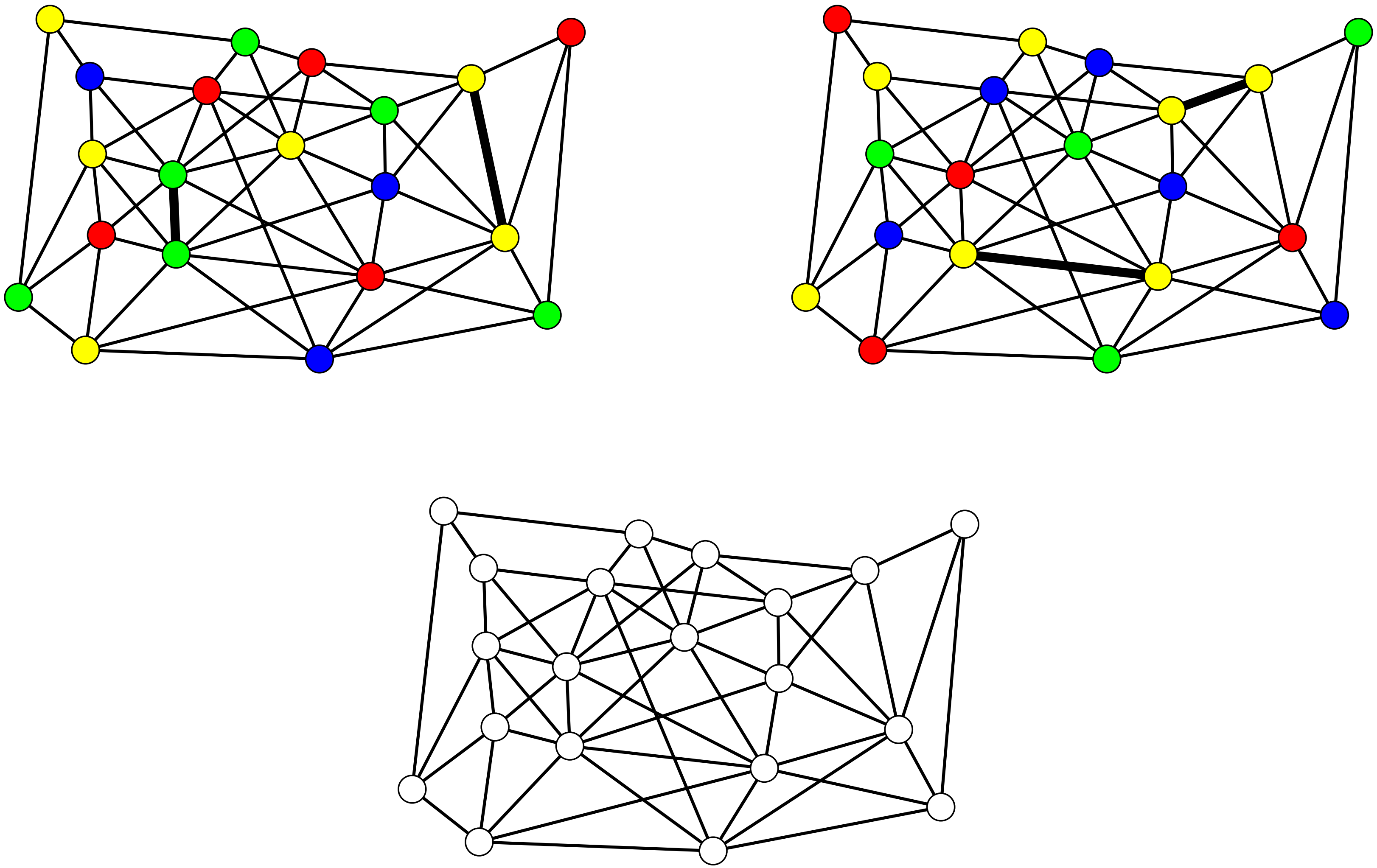
Galinier and Hao Crossover



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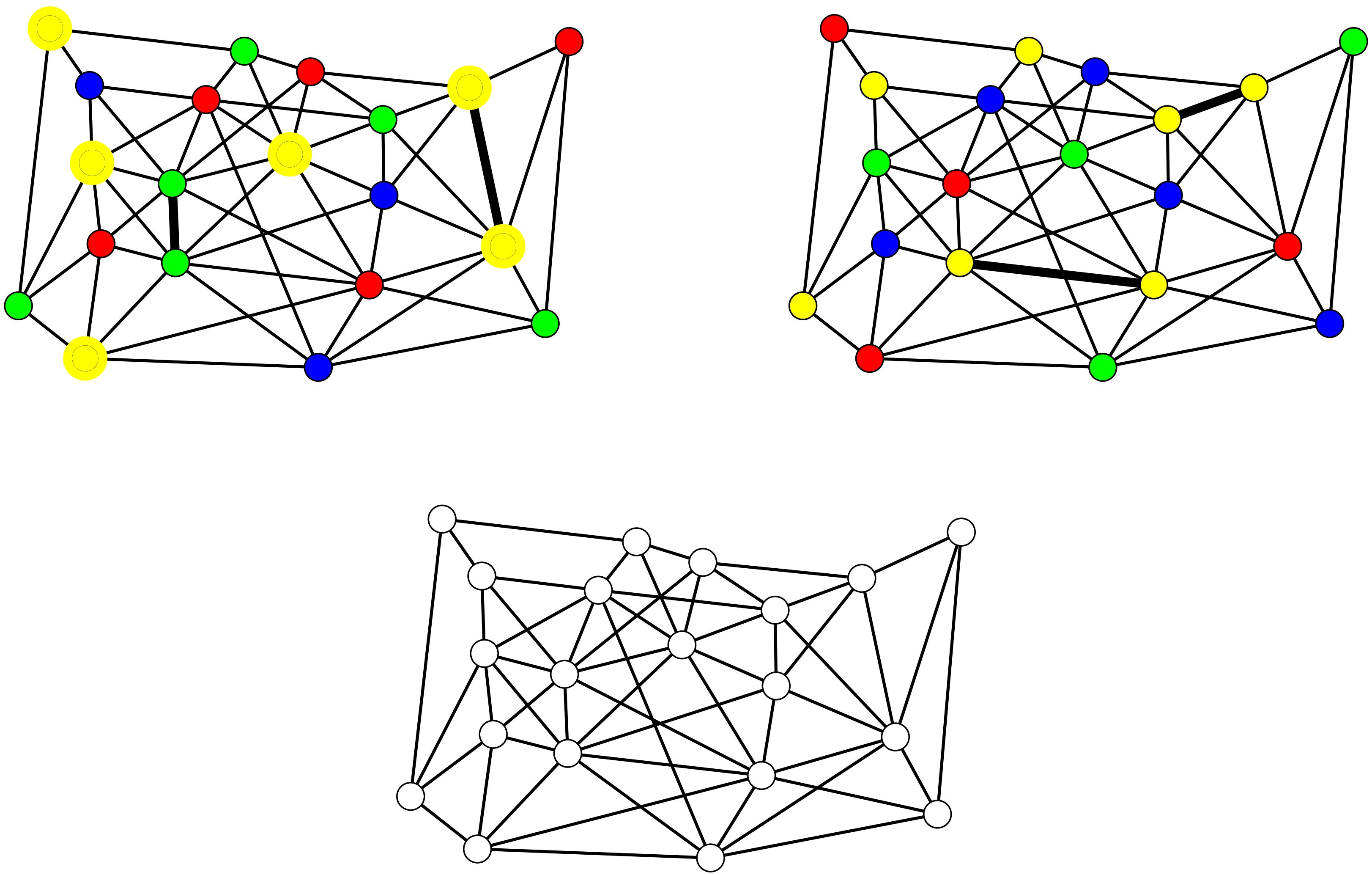

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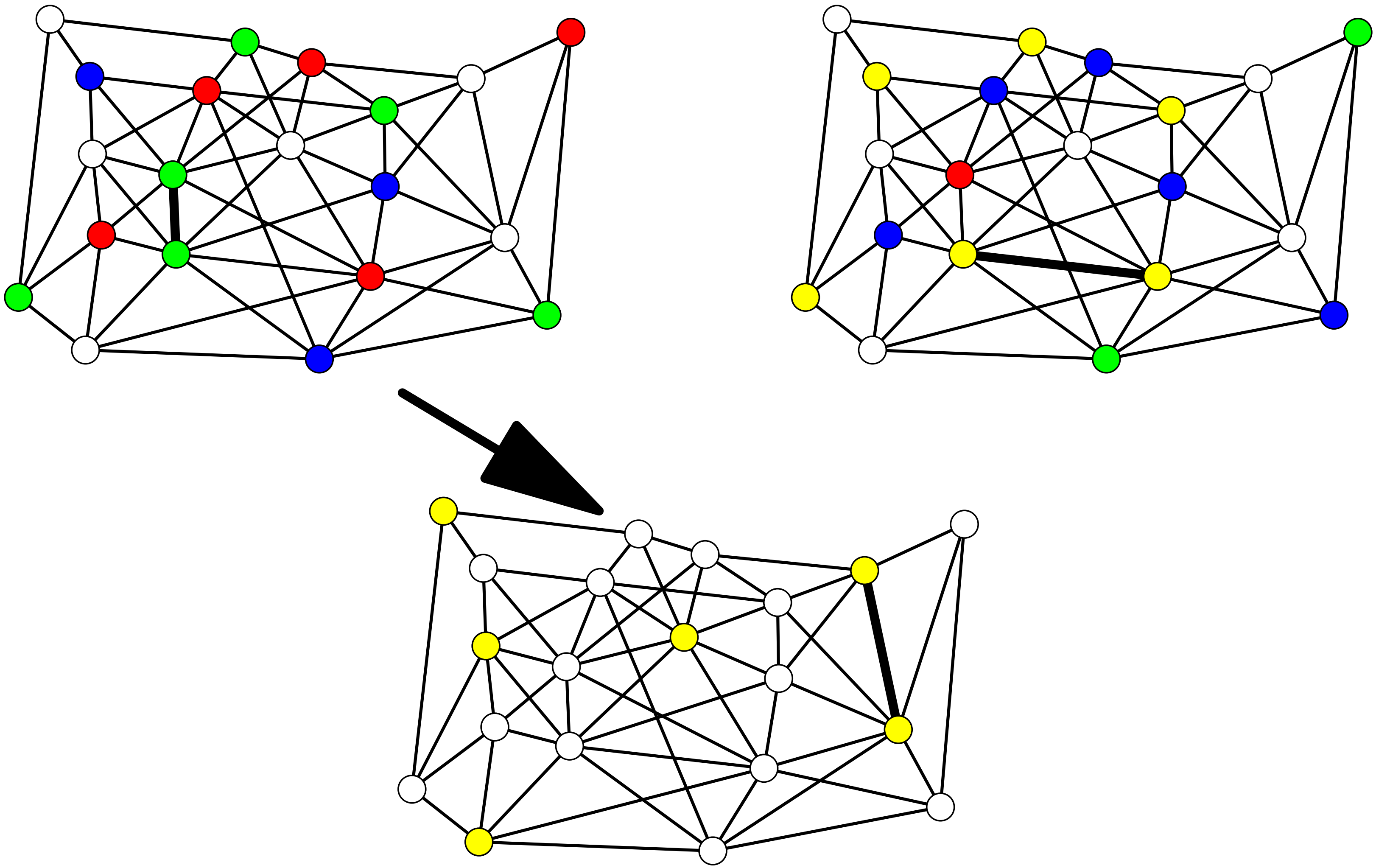
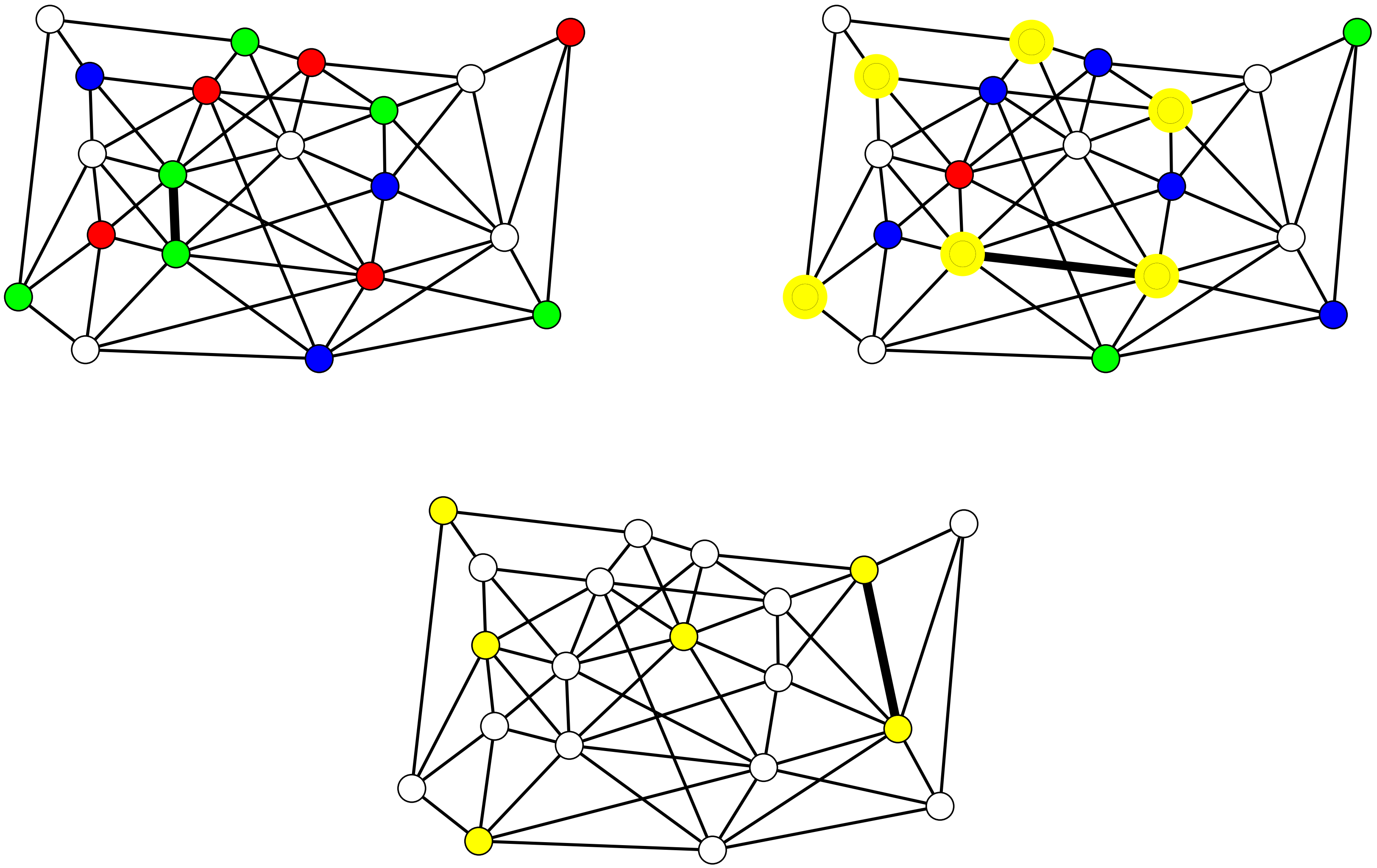


Galinier and Hao Crossover



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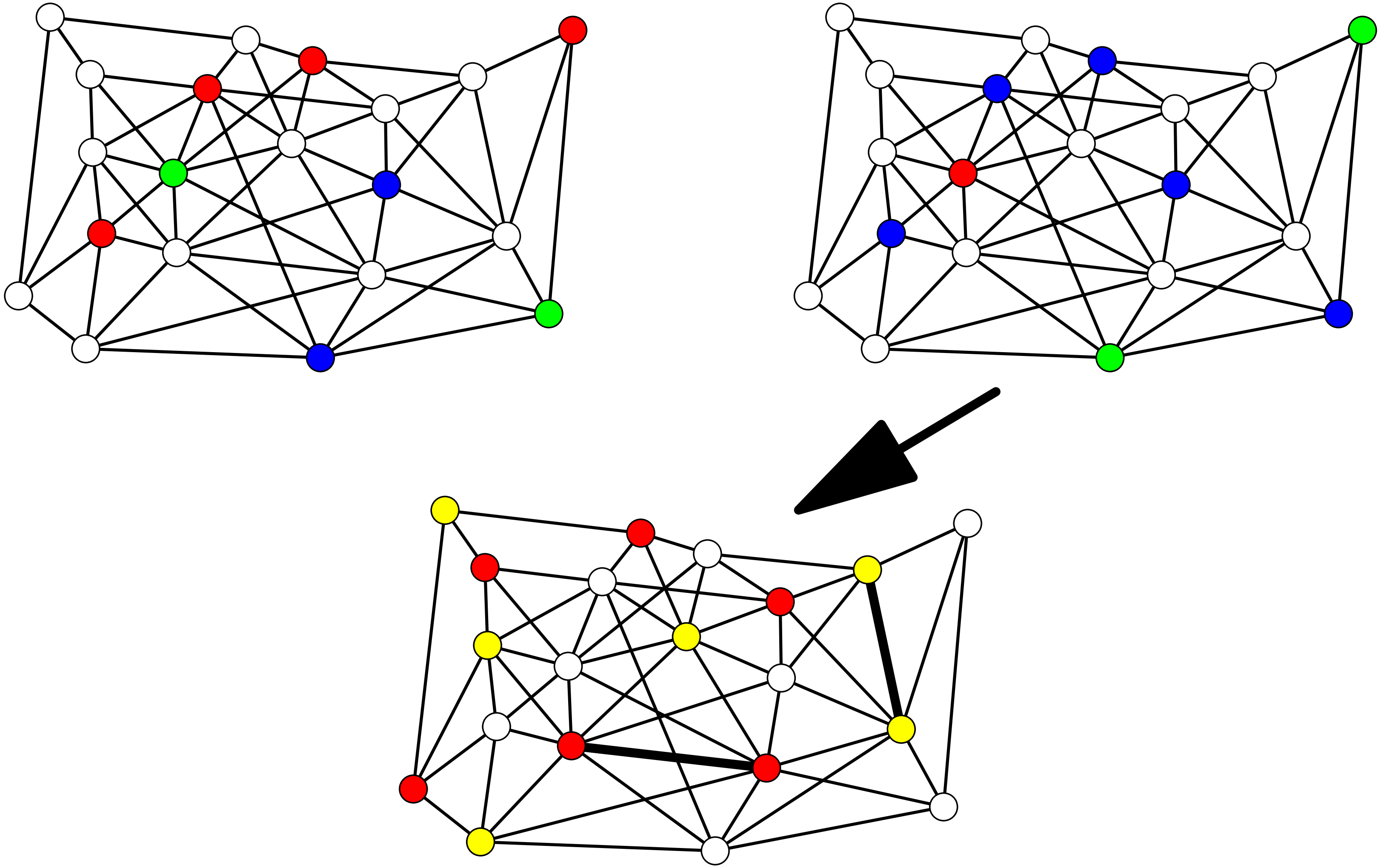

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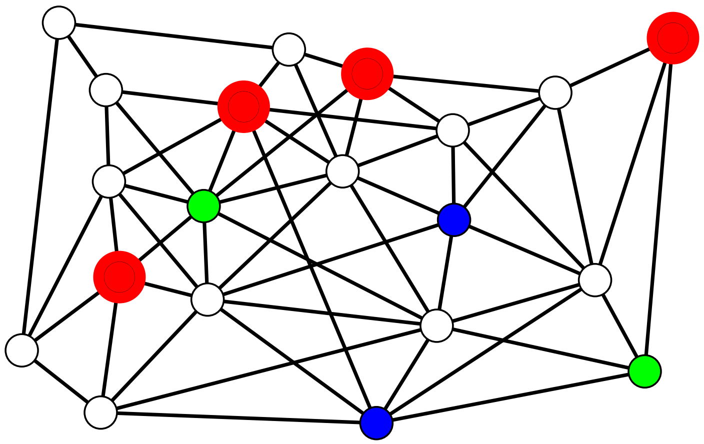

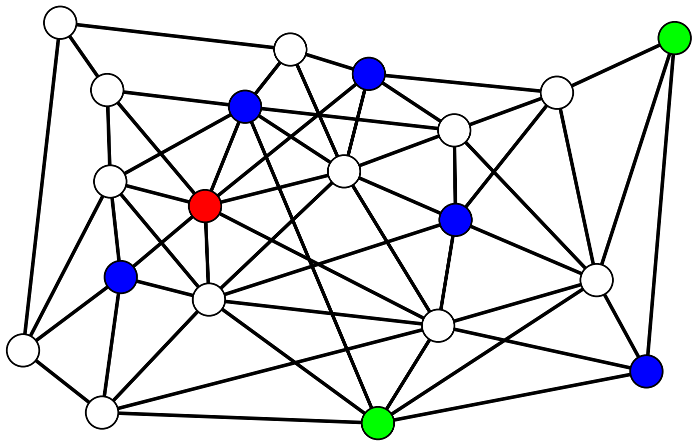

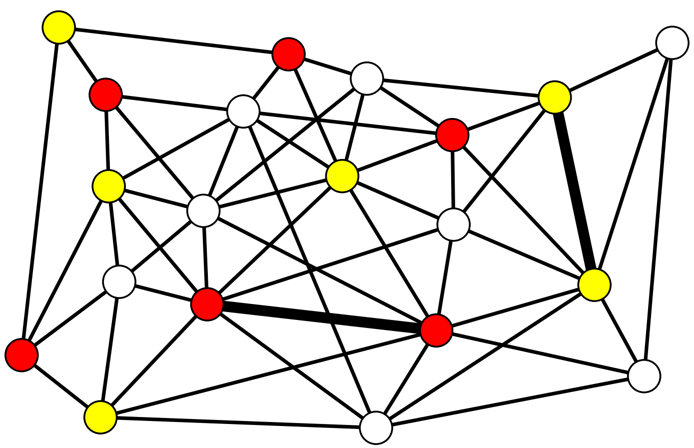

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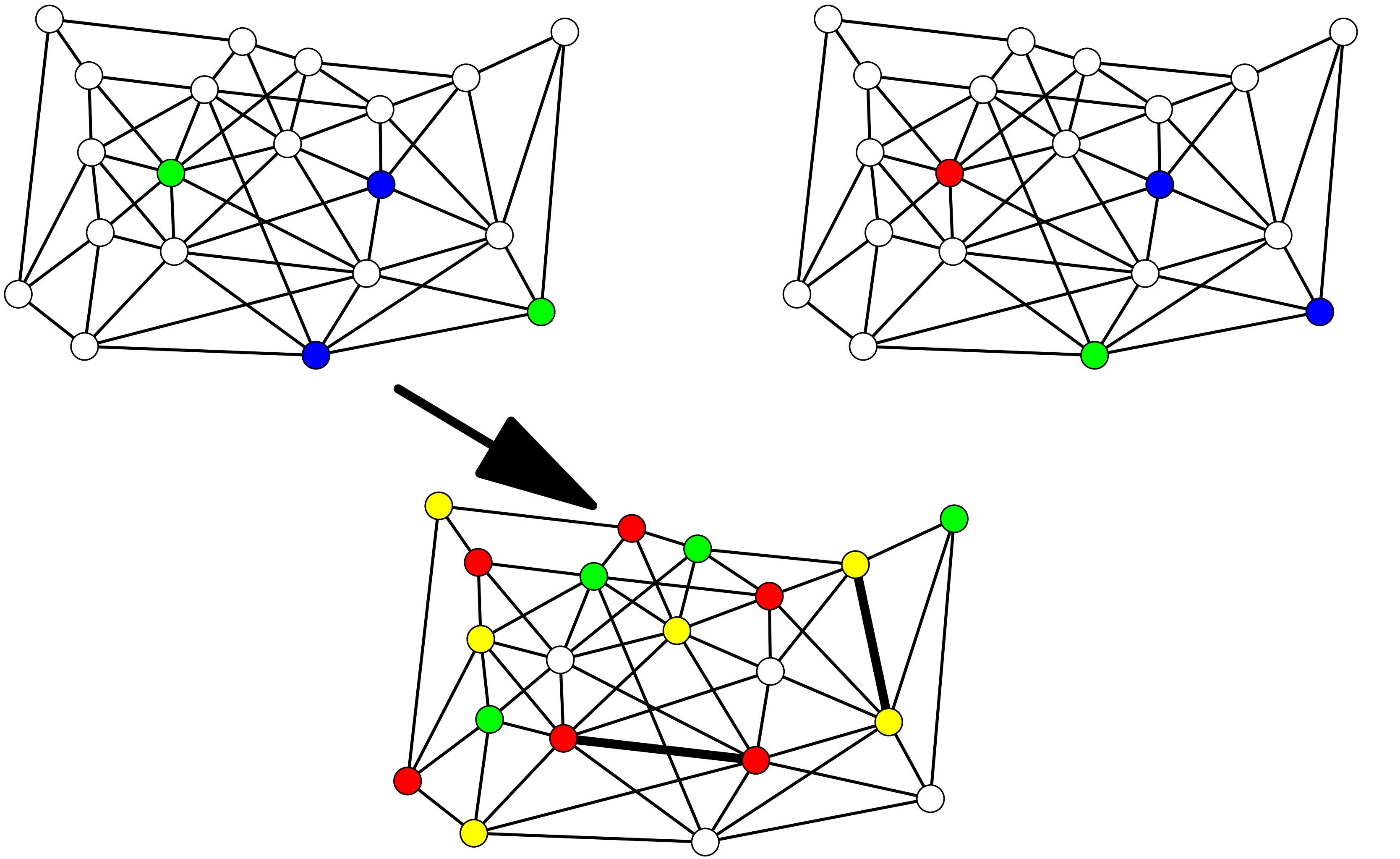

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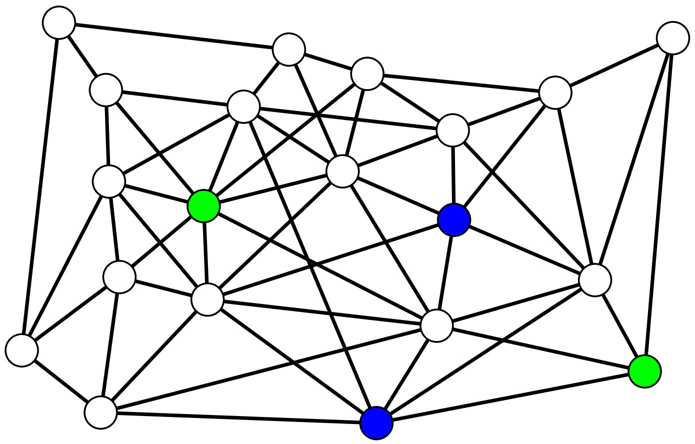

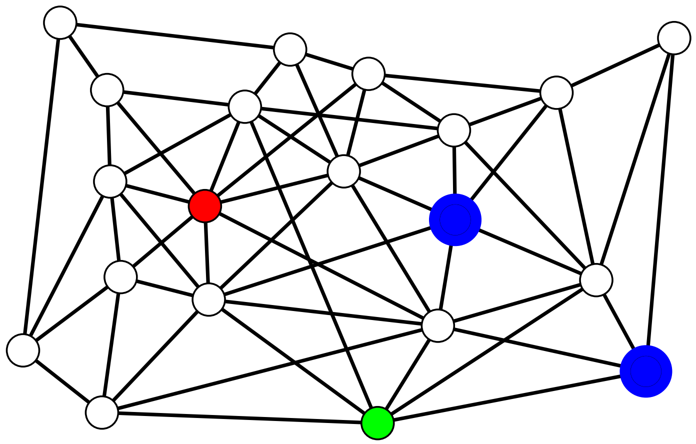

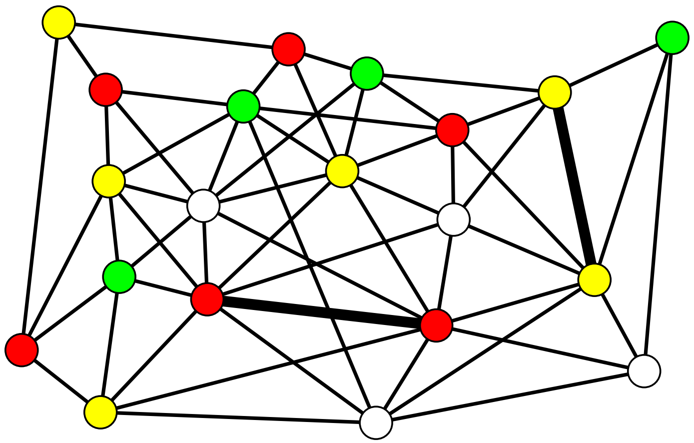

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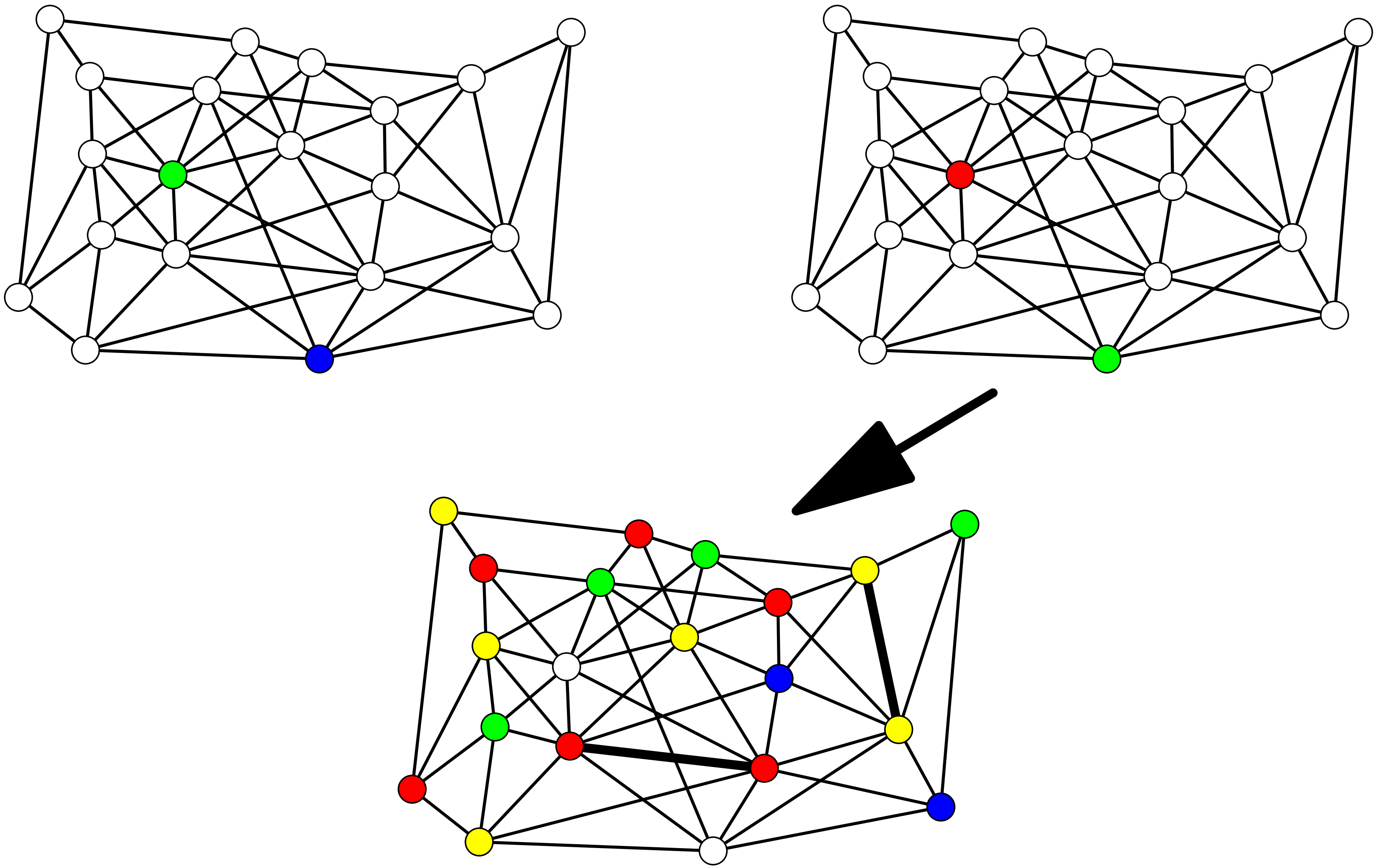

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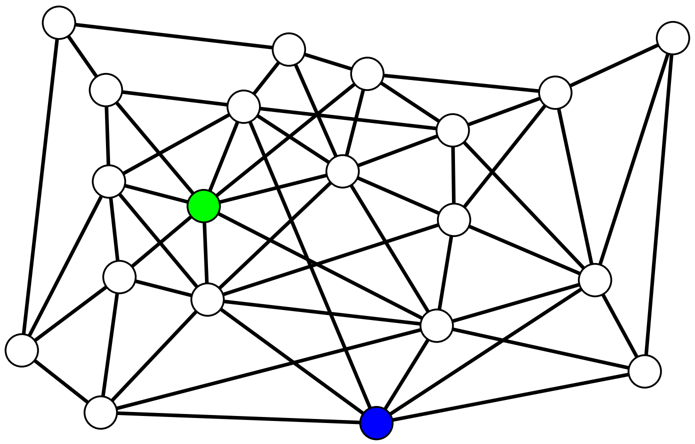

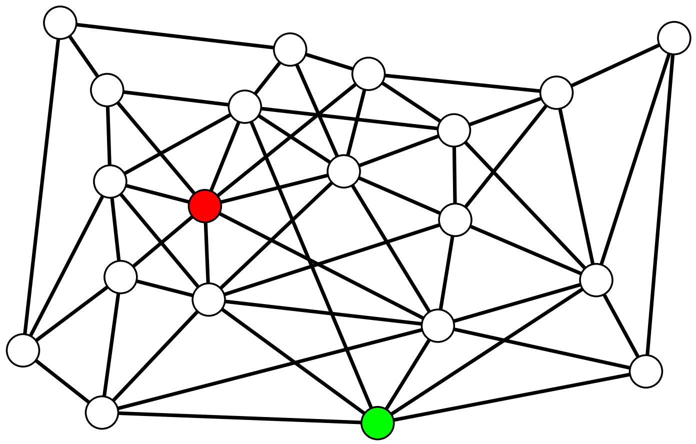

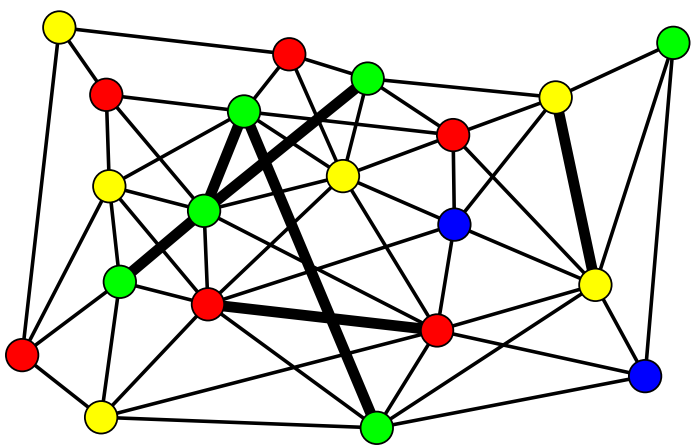

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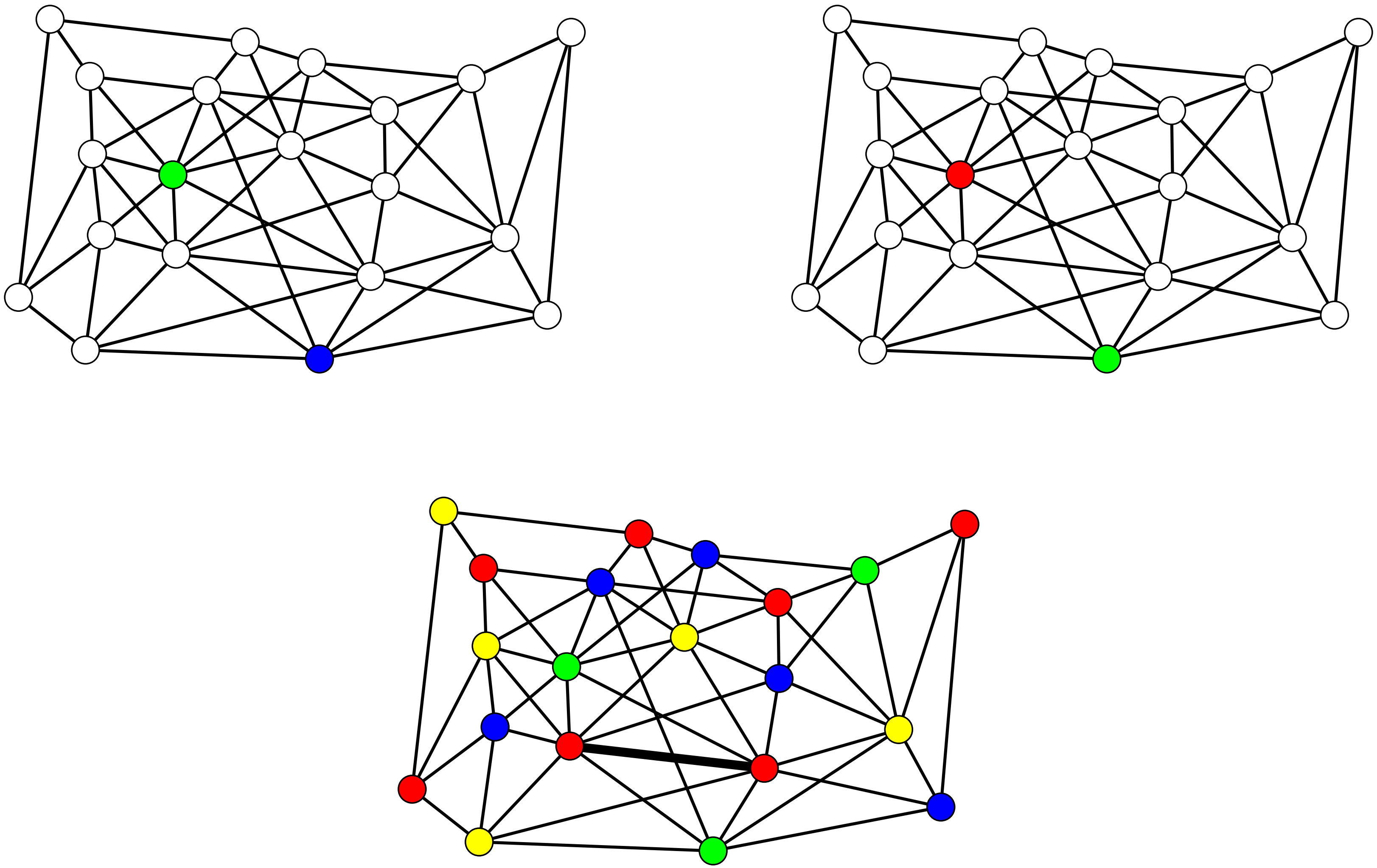

Survey Propagation

- A second example of an algorithm which seems to find good critical variables is survey propagation
- This is a state-of-the-art algorithm applied to SAT and graph colouring (for weakly connected graphs)
- Developed from a statistical mechanics calculation for studying disordered landscapes
- It fixes a few variables at each iteration
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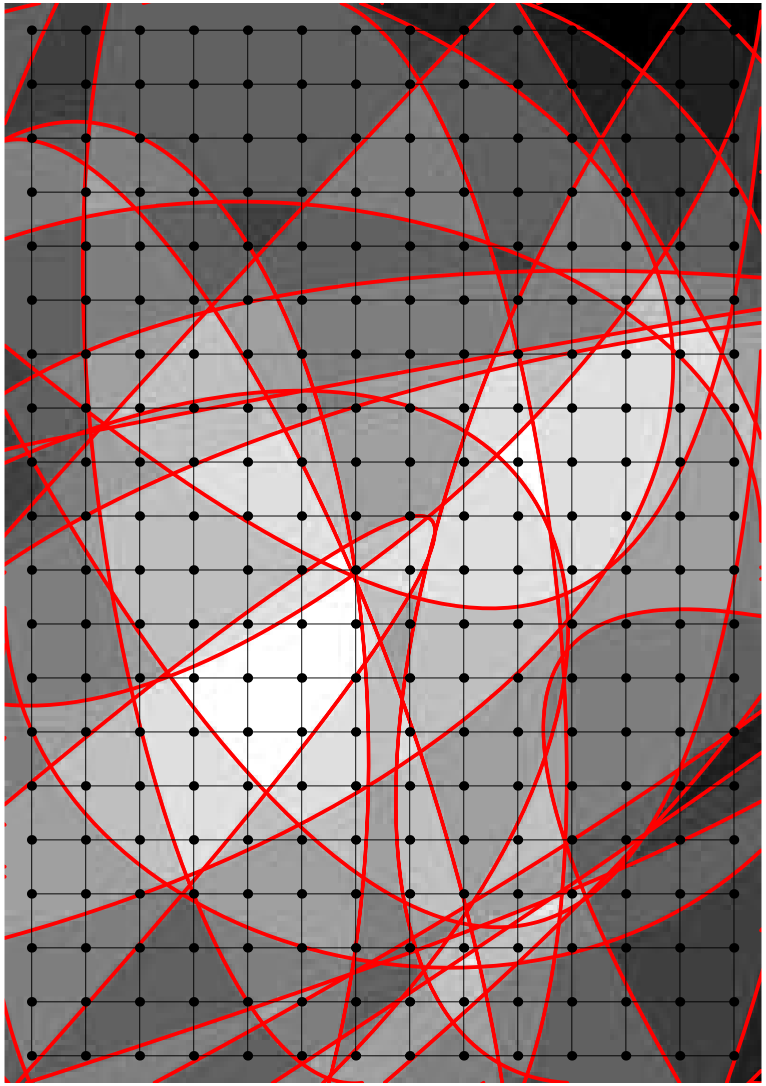
Outline

1. Critical Variable Problems
2. Solving Critical Variable Problems
3. A Toy Example
4. Real Optimisation Problems
5. **Max-Sat**



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Max-Sat

- Problem of choosing binary variables $\boldsymbol{X} \in \{T, F\}^n$ to satisfy as many clauses as possible
- Clause is a disjunction of literals (variables or their negation), e.g.

$$\neg X_4 \vee X_7 \vee \neg X_{17}$$

- Fitness is number of satisfied clauses
- We consider Max-3-Sat where each clause contains 3 literals
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How Max-Sat Becomes Difficult

- For $\alpha \gg \alpha_c$, the problem becomes hard because of the proliferation of local minima
- The better the local optima the larger the basin of attraction
- As the problem becomes large the number of local optima grows exponentially
- This growth is rather slow so small problems are easy to solve by multiple hill-climbing
- Need to look at instances with several hundred variables before they become hard

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- Can we exploit clustering by averaging many solutions?

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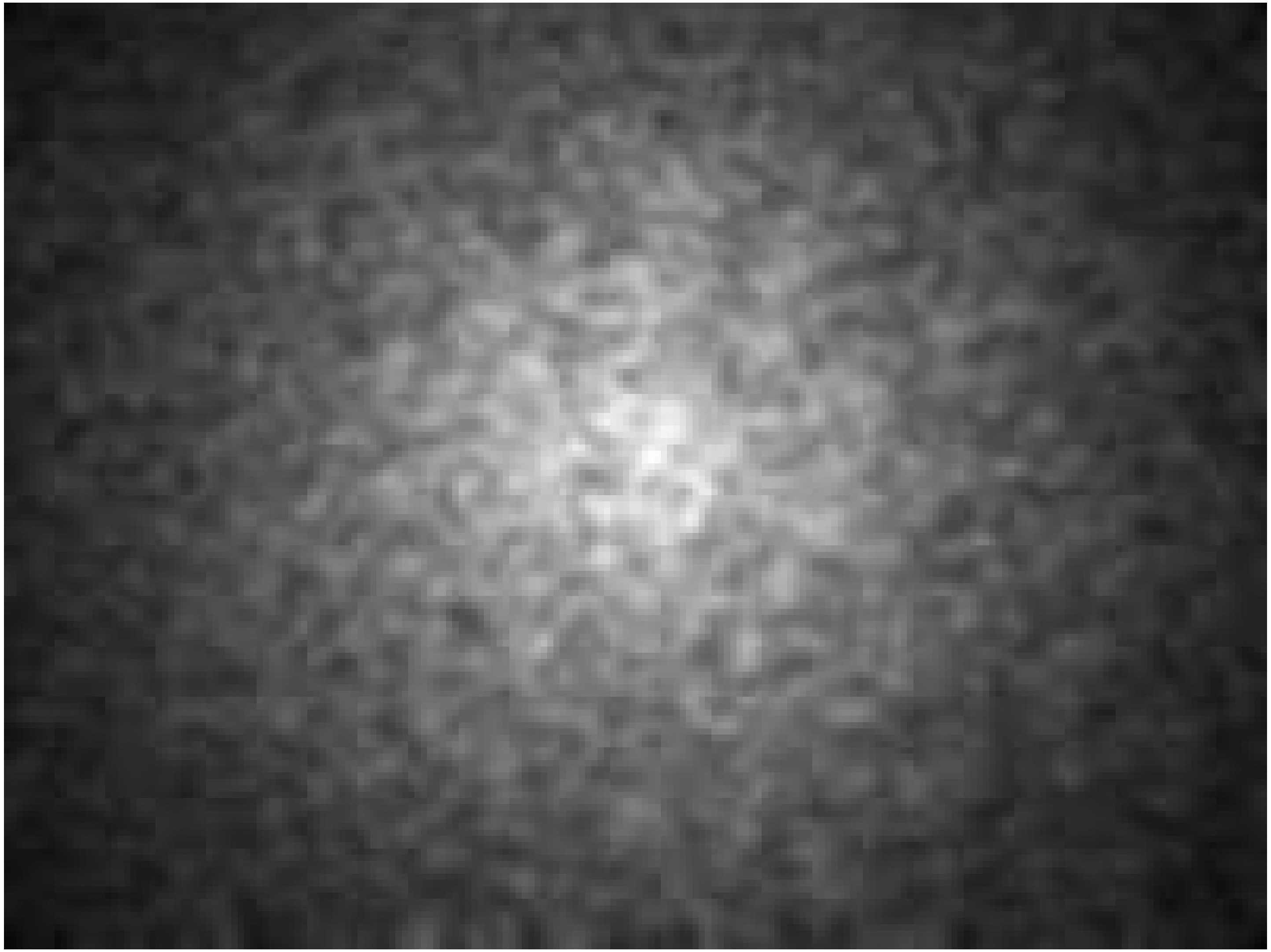

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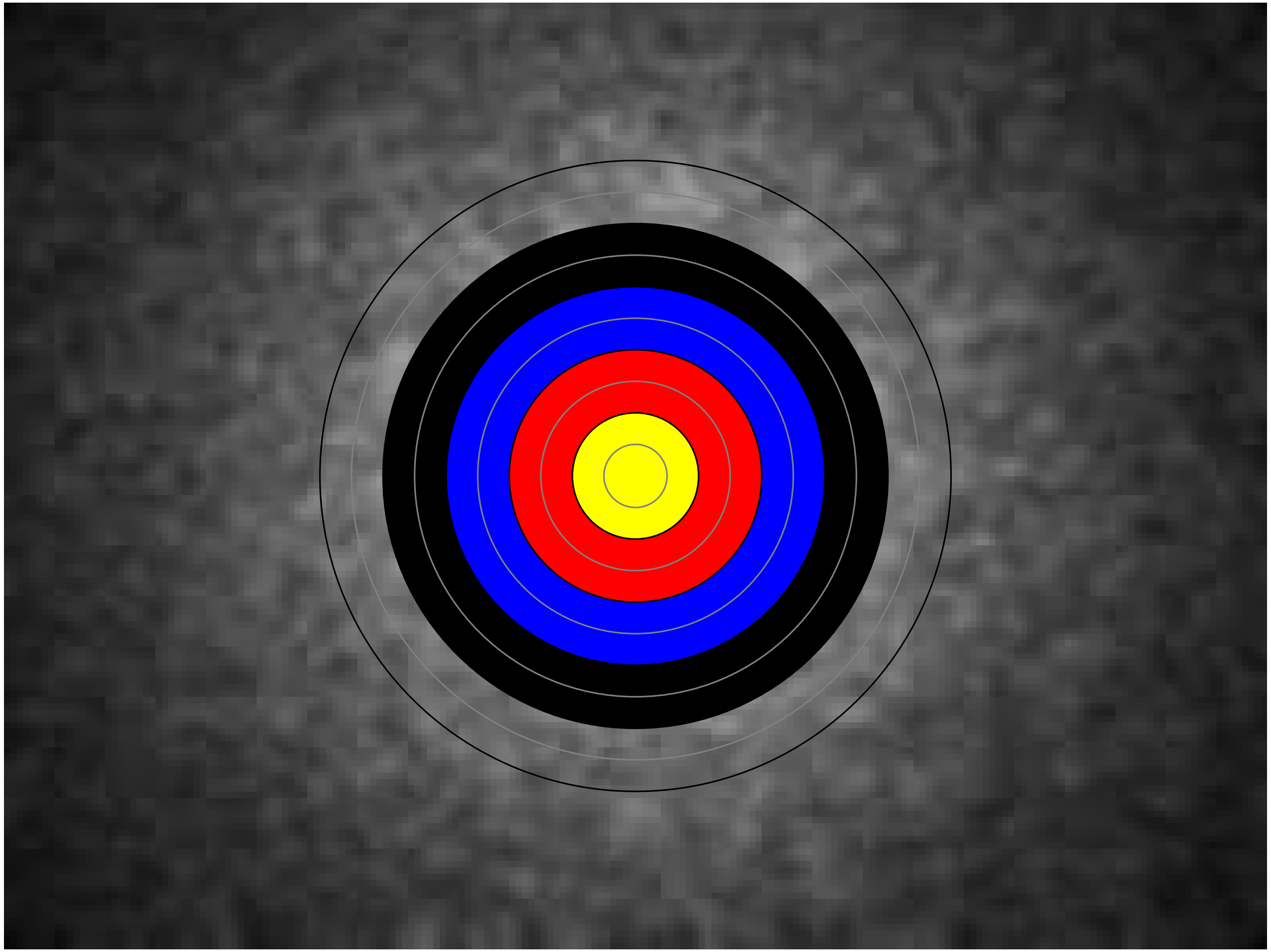

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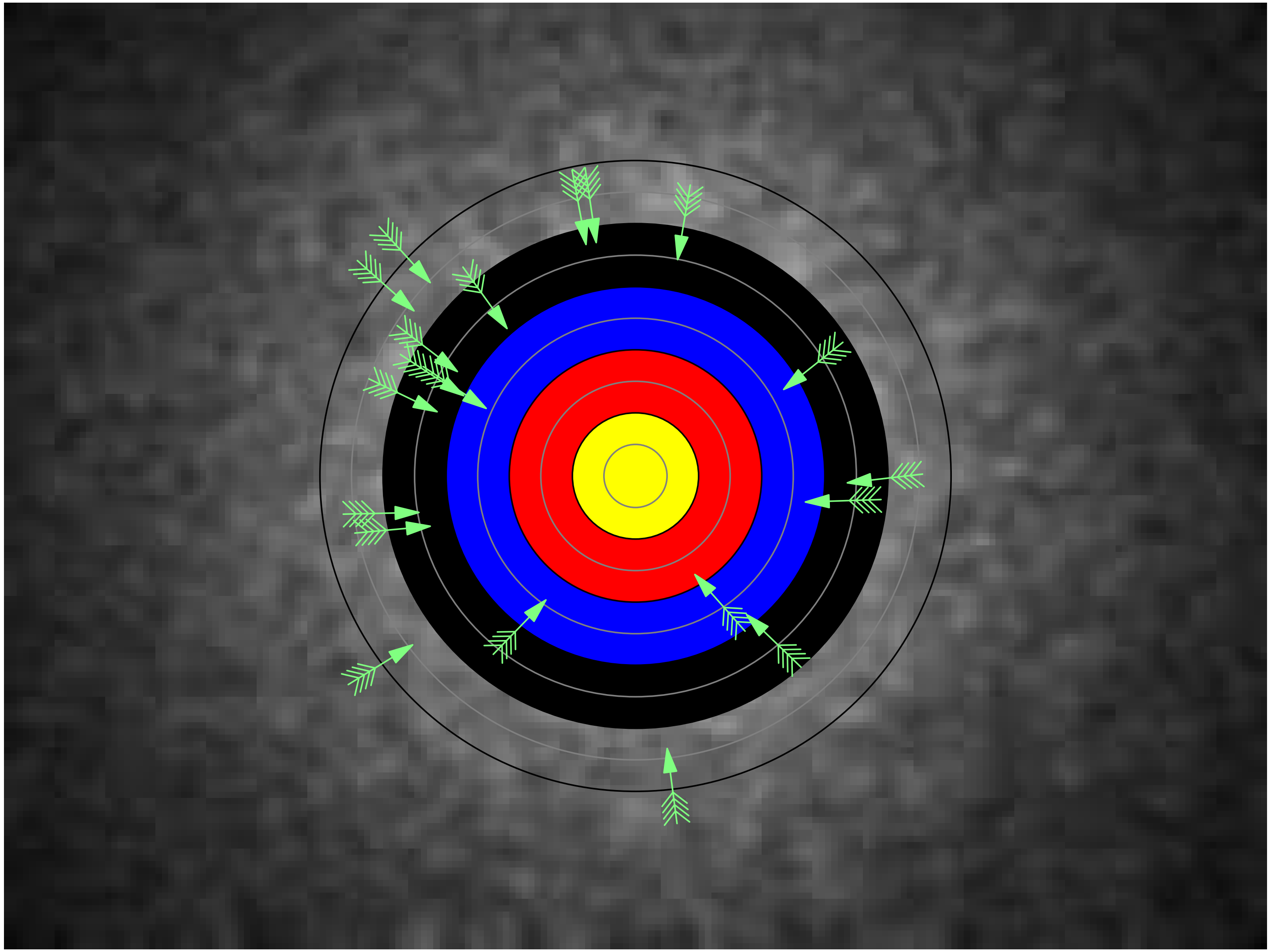

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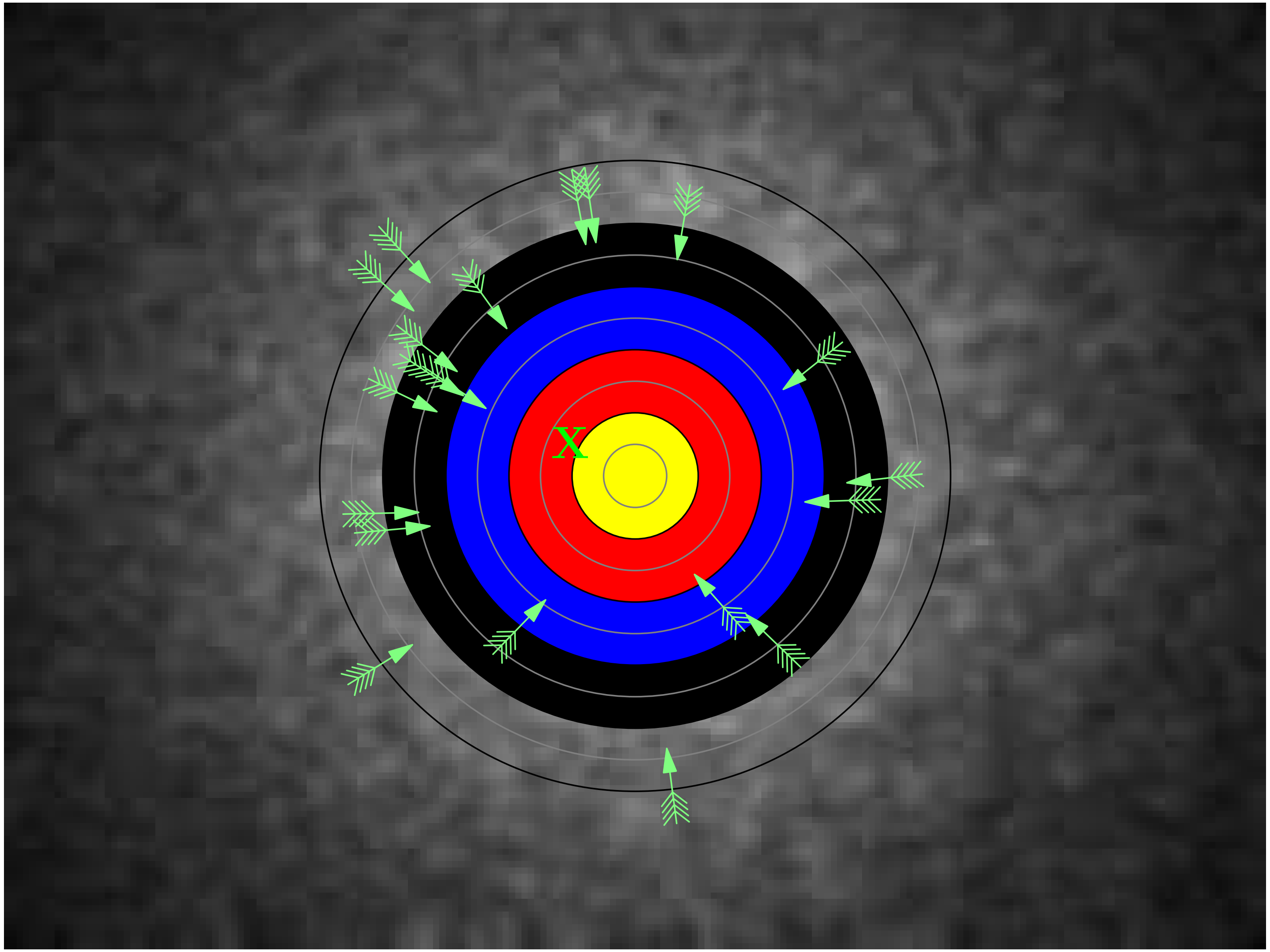

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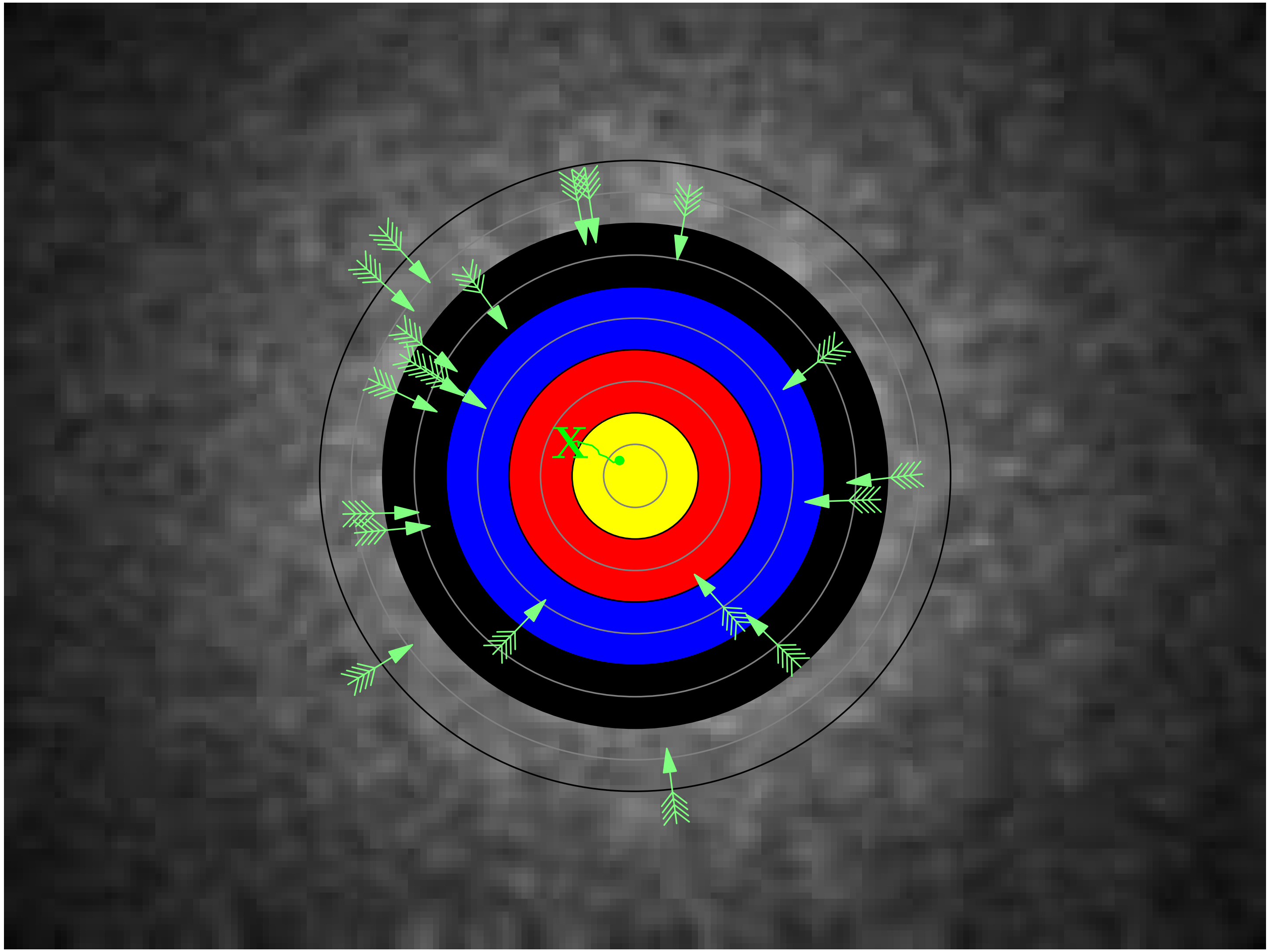

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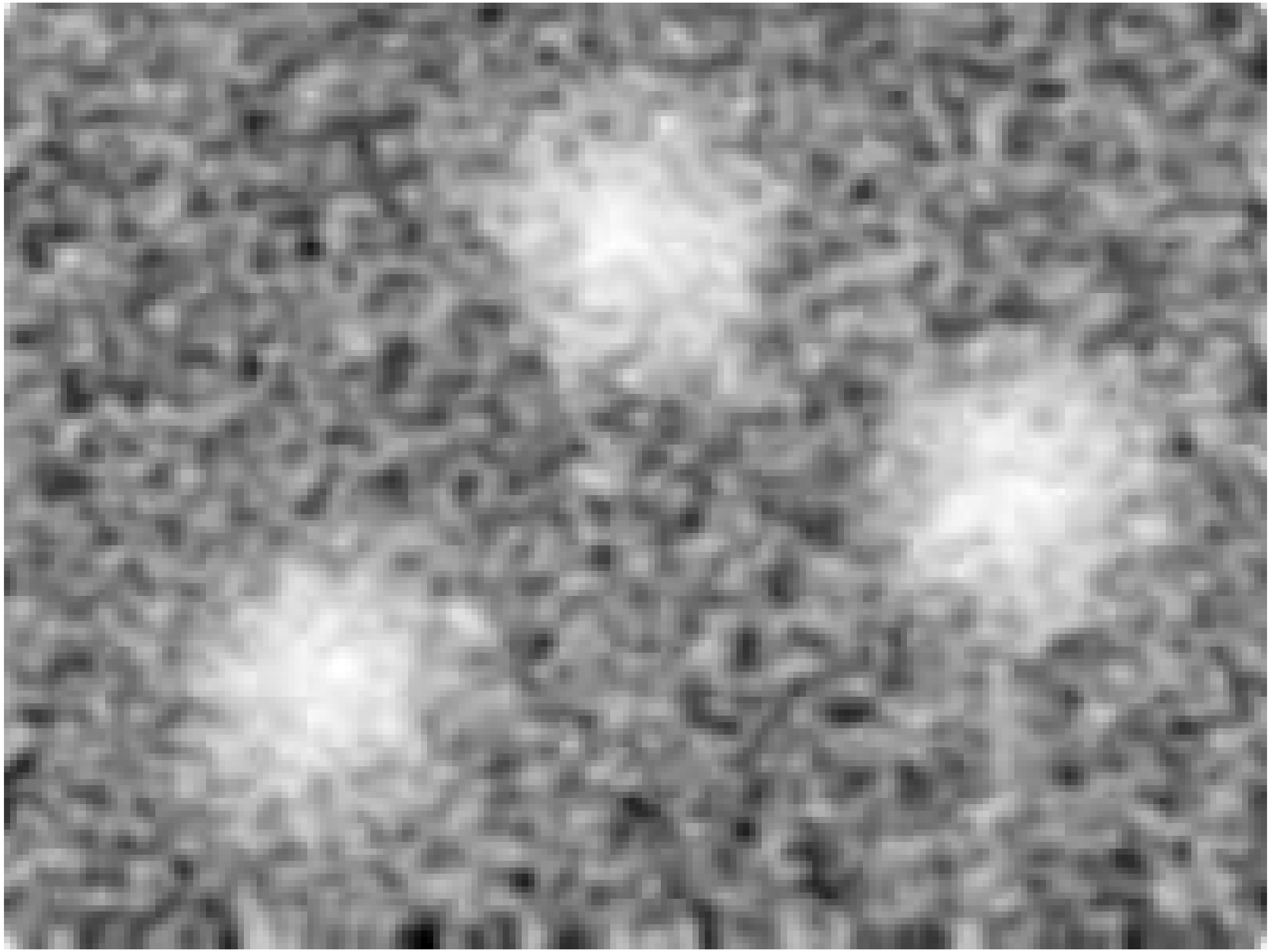

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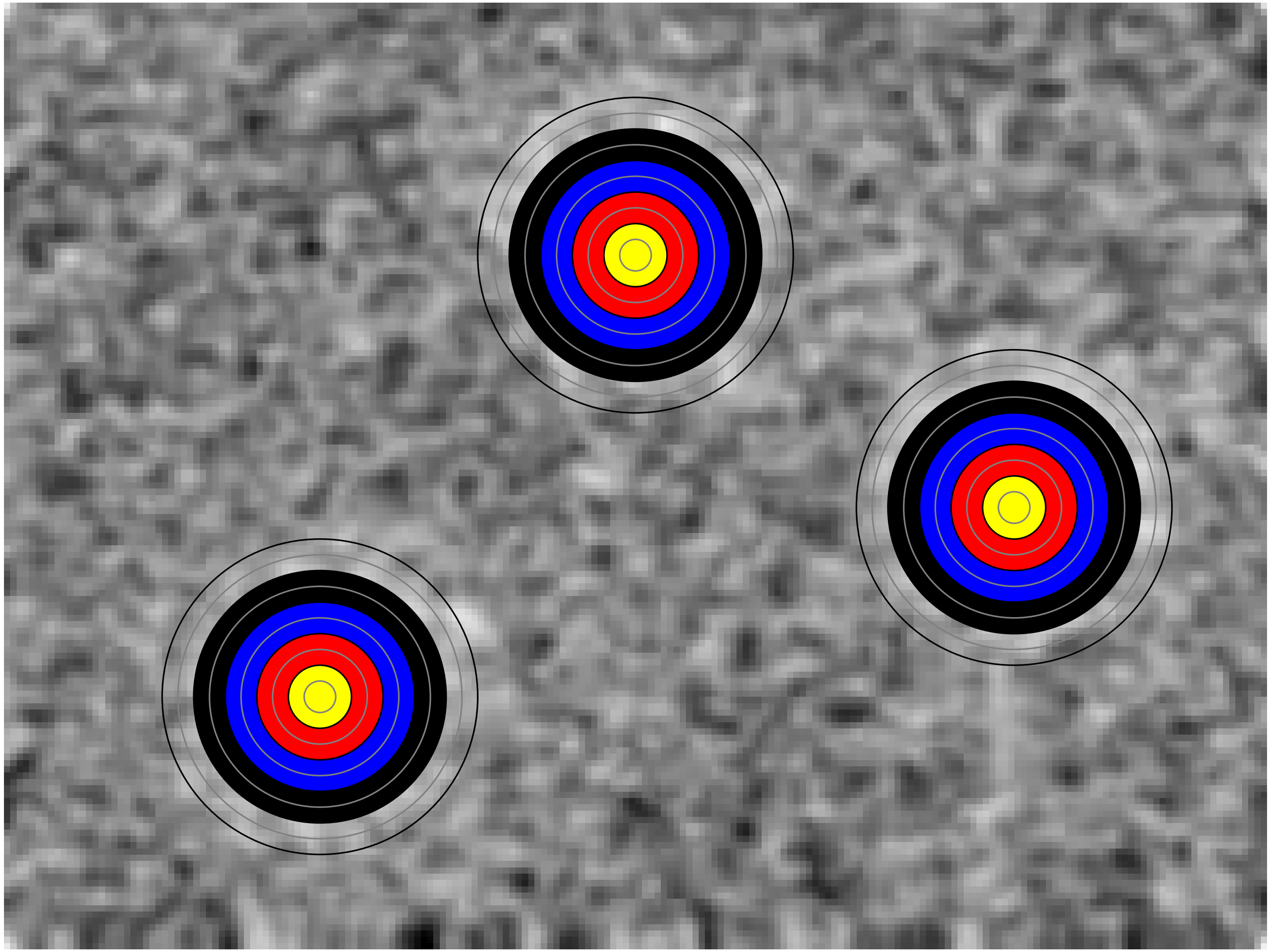

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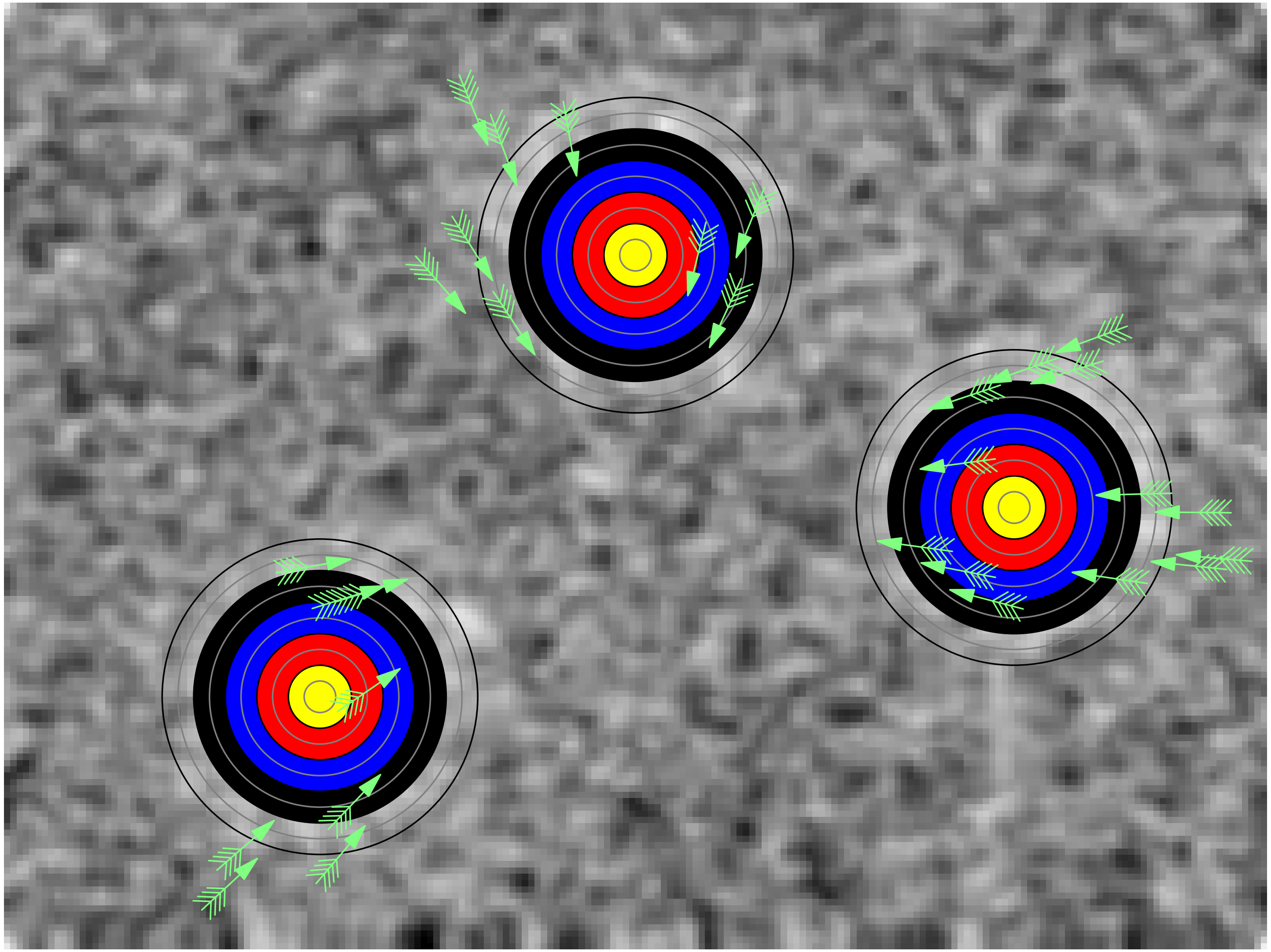

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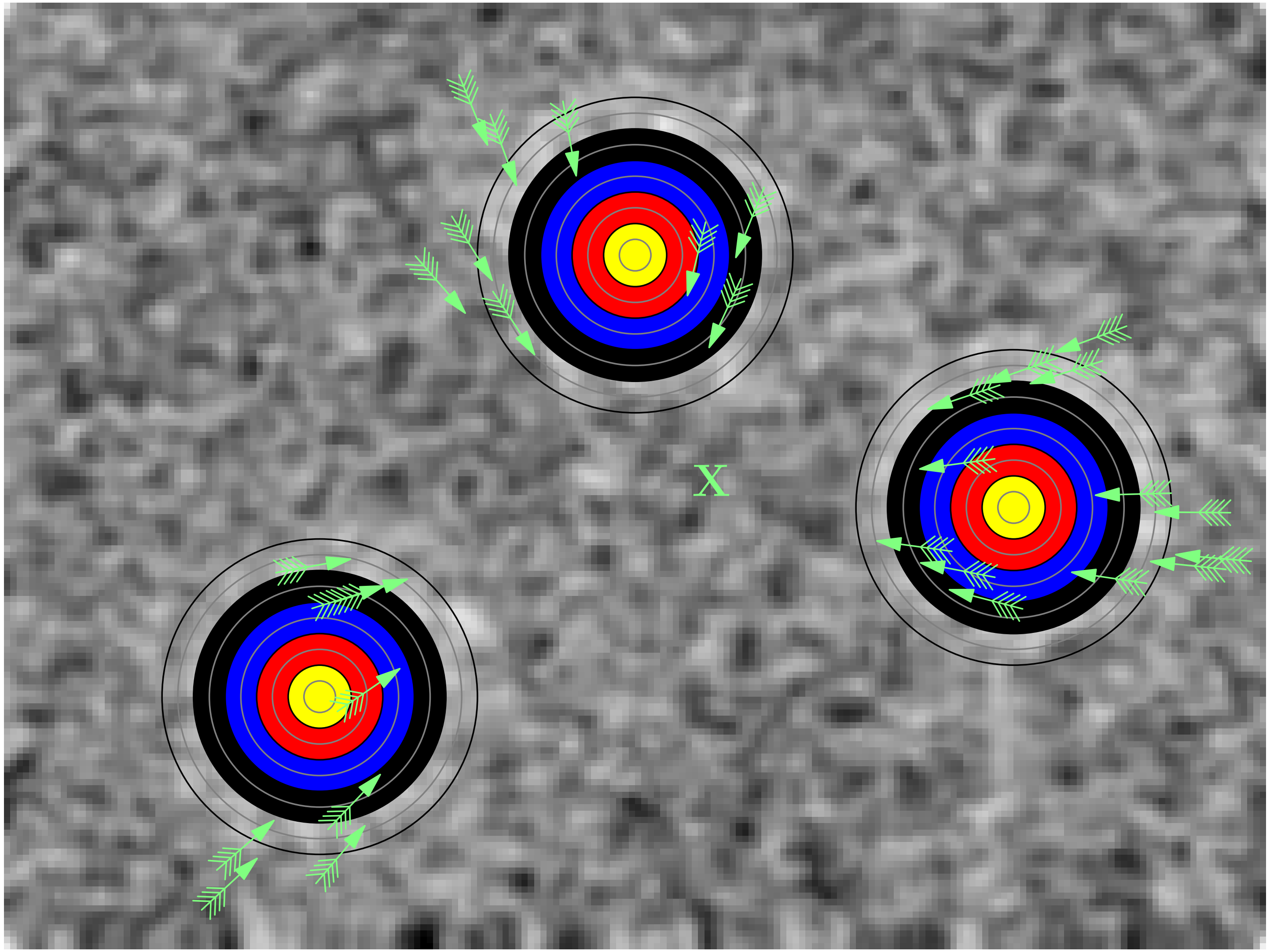

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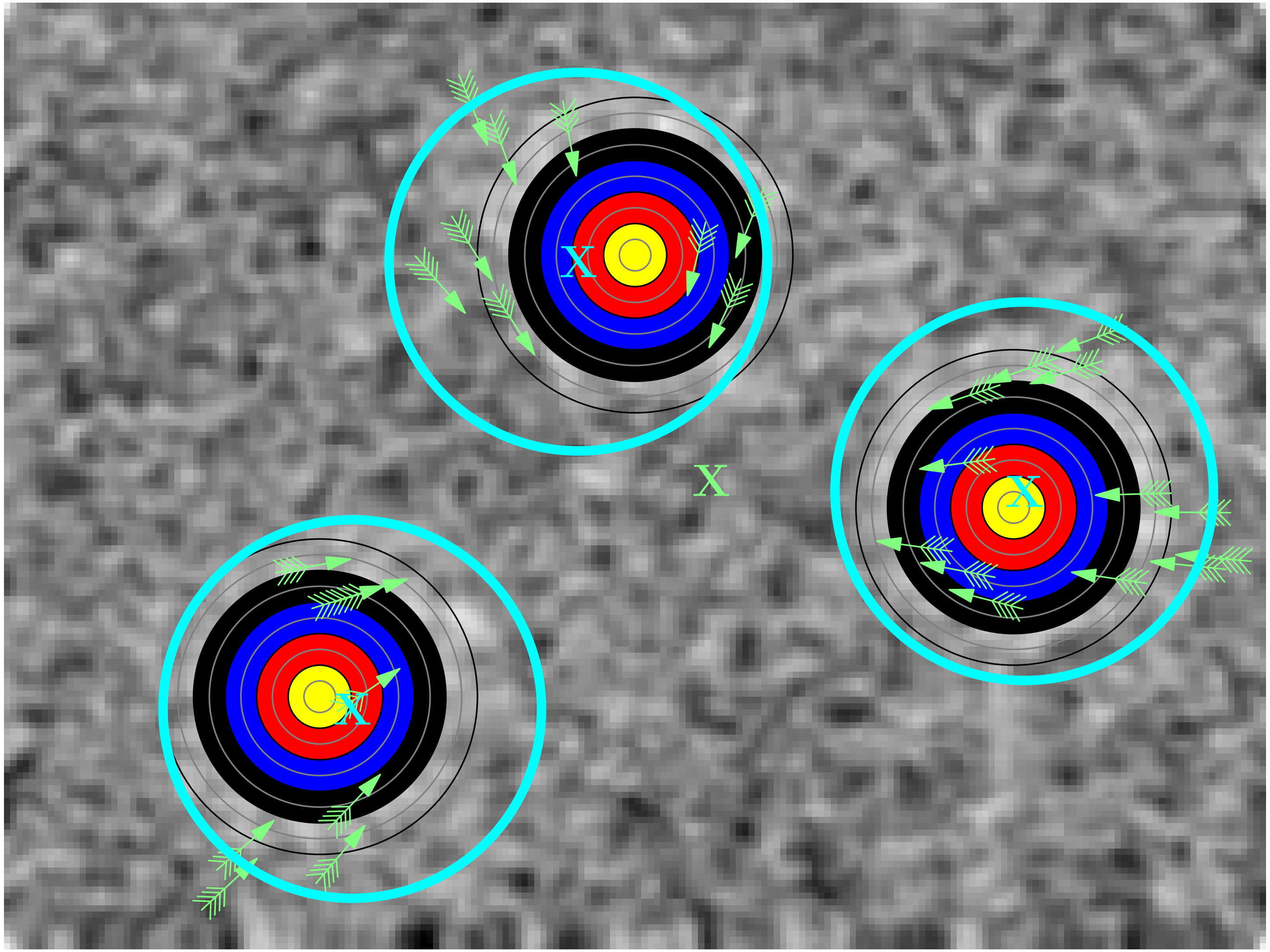

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- In fact, for years it was difficult to even construct a problem where GAs beat hill-climbers (remember the royal road)
- We've learnt how to construct such problems, but they have always seemed implausible as models that occur in practice
- A crazy thought is that maybe populations *really* can solve many real-world problems in a way that other heuristics can't

Any Questions?