Rewriting with Semantic Data Structures: Termination using Dependency Pairs

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Introduction

- Standard TRSs on free data structures cannot model semantic data structures (sets, multisets, sorted lists, sorted arrays, balanced trees, ...)
- We define a class of rewrite systems that allows the use of semantic data structures
- Natural numbers are Built-in, particularly, equality, ordering, and divisibility constraints on natural numbers, and their boolean combinations, are supported to guard when a rewrite rule may be applied
- The rewrite mechanism combines normalized equational rewriting with validity checking of instantiated constraints on natural numbers

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- Expressive and natural framework for writing algorithms
- Termination of program = termination of rewrite system
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Example – Mergesort

Sets are modelled using $\emptyset$, $\langle \cdot \rangle$ (singleton set) and $\cup$. The following algorithm produces a sorted list.

$$
\begin{aligned}
\mathsf{merge}(\mathsf{nil}, y) &\to y \\
\mathsf{merge}(x, \mathsf{nil}) &\to x \\
\mathsf{merge}(\mathsf{cons}(x, xs), \mathsf{cons}(y, ys)) &\to \mathsf{cons}(y, \mathsf{merge}(\mathsf{cons}(x, xs), ys))\ [\![x > y]\!] \\
\mathsf{merge}(\mathsf{cons}(x, xs), \mathsf{cons}(y, ys)) &\to \mathsf{cons}(x, \mathsf{merge}(xs, \mathsf{cons}(y, ys)))\ [\![x \not> y]\!] \\
\mathsf{msort}(\emptyset) &\to \mathsf{nil} \\
\mathsf{msort}(\langle x \rangle) &\to \mathsf{cons}(x, \mathsf{nil}) \\
\mathsf{msort}(x \cup y) &\to \mathsf{merge}(\mathsf{msort}(x), \mathsf{msort}(y))
\end{aligned}
$$

Modelling Semantic Data Structures

- For sets, we expect the following properties:

$$
\begin{aligned}
u \cup v &\approx v \cup u \\
u \cup (v \cup w) &\approx (u \cup v) \cup w \\
u \cup u &\approx u \\
u \cup \emptyset &\approx \emptyset
\end{aligned}
$$

- The mergesort algorithm does not terminate if we use rewriting modulo these properties (or extended rewriting with these properties)
- Solution: Use normalized rewriting (Marché, 1996)
- The last two properties are oriented as rewrite rules and the redex is normalized with these rules before matching it with a rule from $\mathcal{R}$

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Example – Sieve of Eratosthenes

The following algorithm simulates the sieve of Eratosthenes up to a given upper bound.

$$
\begin{array}{rcll}
\mathsf{primes}(x) & \to & \mathsf{sieve}(\mathsf{nats}(2, x)) & \\
\mathsf{nats}(x, y) & \to & \mathsf{nil} & [\![x > y]\!] \\
\mathsf{nats}(x, y) & \to & \mathsf{cons}(x, \mathsf{nats}(x + 1, y)) & [\![x \not> y]\!] \\
\mathsf{sieve}(\mathsf{nil}) & \to & \mathsf{nil} & \\
\mathsf{sieve}(\mathsf{cons}(x, ys)) & \to & \mathsf{cons}(x, \mathsf{sieve}(\mathsf{filter}(x, ys))) & \\
\mathsf{filter}(x, \mathsf{nil}) & \to & \mathsf{nil} & \\
\mathsf{filter}(x, \mathsf{cons}(y, zs)) & \to & \mathsf{cond}(\mathsf{isdiv}(x, y), x, y, zs) & \\
\mathsf{cond}(\mathsf{true}, x, y, zs) & \to & \mathsf{filter}(x, zs) & \\
\mathsf{cond}(\mathsf{false}, x, y, zs) & \to & \mathsf{cons}(y, \mathsf{filter}(x, zs)) & \\
\mathsf{isdiv}(x, 0) & \to & \mathsf{true} & [\![x > 0]\!] \\
\mathsf{isdiv}(x, y) & \to & \mathsf{false} & [\![x > y \wedge y > 0]\!] \\
\mathsf{isdiv}(x, x + y) & \to & \mathsf{isdiv}(x, y) & [\![x > 0]\!]
\end{array}
$$

Using $\mathcal{PA}$-constraints

- We use the following properties of natural numbers:

$$\begin{aligned} u + v &\approx v + u \\ u + (v + w) &\approx (u + v) + w \\ u + 0 &\approx u \end{aligned}$$

- $\mathcal{PA}$-constraints are Boolean combinations of atomic $\mathcal{PA}$-constraints of the form
  - $s \simeq t$,
  - $s > t$, and
  - $k \mid s$ (for some $k \in \mathbb{N} - \{0\}$)
- $s \geq t$, $s \not> t$, $s \not\simeq t$, ... can be defined as usual
- A rewrite rule can only be applied if the constraint becomes $\mathcal{PA}$-valid after being instantiated with the matching substitution

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Example – Imperative Program 1

- Consider the imperative program fragment (Cook *et. al.*, 2005)

```
while (x > 0) {
    x++;
    y := 1;
    while (x > y) {
        y++
    }
    x := x - 2
}
```

- It can be translated into

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\begin{array}{rcll}
\mathsf{eval}_0(x, y) & \to & \mathsf{eval}_1(x+1, 1) & [\![x > 0]\!] \\
\mathsf{eval}_1(x, y) & \to & \mathsf{eval}_1(x, y+1) & [\![x > y]\!] \\
\mathsf{eval}_1(x+2, y) & \to & \mathsf{eval}_0(x, y) & [\![x+2 \not> y]\!]
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\mathsf{eval}_1(x + 2, y) & \to & \mathsf{eval}_0(x, y) & [\![x + 2 \not> y]\!]
\end{array}
$$

Example – Imperative Program 2

- Consider the imperative program fragment

```
while (x > 0 || y > 0) {
    if (x > 0) {
        x--
    } else if (y > 0) {
        y--
    } else {
        continue
    }
}
```

- It can be translated into

$$\begin{aligned} \mathsf{eval}(x+1,y) &\to \mathsf{eval}(x,y) \ [\![(x+1>0 \lor y>0) \land x+1>0]\!] \\ \mathsf{eval}(x,y+1) &\to \mathsf{eval}(x,y) \ [\![(x>0 \lor y+1>0) \land x \not> 0 \land y+1>0]\!] \\ \mathsf{eval}(x,y) &\to \mathsf{eval}(x,y) \ [\![(x>0 \lor y>0) \land x \not> 0 \land y \not> 0]\!] \end{aligned}$$

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\mathsf{eval}(x, y) &\to \mathsf{eval}(x, y)\ [\![(x > 0 \vee y > 0) \wedge x \not> 0 \wedge y \not> 0]\!]
\end{aligned}$$

Constrained Equational Systems

- A constrained equational system (CES) has the form $(\mathcal{R}, \mathcal{S}, \mathcal{E})$ where
  - $\mathcal{R}$ is a finite set of rewrite rules with $\mathcal{PA}$-constraints,
  - $\mathcal{S}$ is a finite set of rewrite rules with $\mathcal{PA}$-constraints over the constructors of $\mathcal{R}$,
  - $\mathcal{E}$ is a finite set of equations over the constructors of $\mathcal{R}$
- $\mathcal{S}$ and $\mathcal{E}$ model the semantic data structures
- Assumptions are imposed on $\mathcal{S}$ and $\mathcal{E}$
- We assume that no function symbol has sort $\mathtt{nat}$
- Rewriting with a CES: $s \xrightarrow{\mathcal{S}}_{\mathcal{PA}\|\mathcal{E}\backslash\mathcal{R}} t$ iff
  - $s|_p \downarrow_{\mathcal{PA}\|\mathcal{E}\backslash\mathcal{S}} \sim_{\mathcal{E}\cup\mathcal{PA}} l\sigma$,
  - $C\sigma$ is $\mathcal{PA}$-valid, and
  - $t = s[r\sigma]_p$

  for some position p, some substitution σ, and some rule $l \to r[\![C]\!]$ from $\mathcal{R}$ (Here, $\downarrow_{\mathcal{PA}\|\mathcal{E}\backslash\mathcal{S}}$ means normalization w.r.t. $\mathcal{S}$)

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Rewrite Relation Illustrated

$$\mathcal{E}: \quad u \cup v \approx v \cup u \qquad u \cup (v \cup w) \approx (u \cup v) \cup w$$

$$\mathcal{S}: \quad u \cup u \to u \qquad u \cup \emptyset \to u$$

- Reduce $t = \mathsf{msort}(\langle 1 \rangle \cup (\langle 3 \rangle \cup \langle 1 \rangle))$ using $\xrightarrow{\mathcal{S}}_{\mathcal{PA}\|\mathcal{E}\backslash\mathcal{R}}$
- Use $\mathsf{msort}(x \cup y) \to \mathsf{merge}(\mathsf{msort}(x), \mathsf{msort}(y))$ with $\sigma = \{x \mapsto \langle 1 \rangle, y \mapsto \langle 3 \rangle\}$
- $t \xrightarrow{!}{}^{<\varepsilon}_{\mathcal{PA}\|\mathcal{E}\backslash\mathcal{S}} \mathsf{msort}(\langle 1 \rangle \cup \langle 3 \rangle) \sim^{<\varepsilon}_{\mathcal{E}\cup\mathcal{PA}} \mathsf{msort}(x \cup y)\sigma$
- Thus $t \xrightarrow{\mathcal{S}}_{\mathcal{PA}\|\mathcal{E}\backslash\mathcal{R}} \mathsf{merge}(\mathsf{msort}(\langle 1 \rangle), \mathsf{msort}(\langle 3 \rangle)) = t'$
- $t' \xrightarrow{\mathcal{S}}{}^{+}_{\mathcal{PA}\|\mathcal{E}\backslash\mathcal{R}} \mathsf{merge}(\mathsf{cons}(1, \mathsf{nil}), \mathsf{cons}(3, \mathsf{nil}))$
- $\mathsf{merge}(\mathsf{cons}(x, xs), \mathsf{cons}(y, ys)) \to \mathsf{cons}(x, \mathsf{merge}(xs, \mathsf{cons}(y, ys)))[\![x \not> y]\!]$ with $\sigma = \{x \mapsto 1, xs \mapsto \mathsf{nil}, y \mapsto 3, ys \mapsto \mathsf{nil}\}$
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$\mathcal{E}$: $u \cup v \approx v \cup u$
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- Virtually all automated termination tools for rewriting use dependency pairs
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Our Contributions

- Extension to rewriting with semantic data structures and built-in natural numbers, i.e., constraints on natural numbers are allowed
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Key Result

- As usual, a dependency pair corresponds to a call to a defined function:
  $\mathrm{DP}(\mathcal{R}) = \{ l^\sharp \to t^\sharp [\![C]\!] \mid l \to r [\![C]\!] \in \mathcal{R},\ t$ is a subterm of r with defined root symbol $\}$
- Note that the dependency pairs inherit the constraints
- A sequence $s_1 \to t_1 [\![C_1]\!], s_2 \to t_2 [\![C_2]\!], \ldots$ of dependency pairs from a set $\mathcal{P}$ is a $(\mathcal{P}, \mathcal{R})$-chain iff there exists a σ such that

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Example

- Sets are modelled using $\emptyset$ and ins
- $\mathsf{nats}(x, y)$ computes $\{z \mid x \leq z \leq y\}$
- $$\begin{array}{llcll} \mathcal{E}: & \mathsf{ins}(x, \mathsf{ins}(y, zs)) & \approx & \mathsf{ins}(y, \mathsf{ins}(x, zs)) & \\ \mathcal{S}: & \mathsf{ins}(x, \mathsf{ins}(y, zs)) & \to & \mathsf{ins}(x, ys) & [\![x \simeq y]\!] \\ \mathcal{R}: & \mathsf{inc}(\emptyset) & \to & \emptyset & \\ & \mathsf{inc}(\mathsf{ins}(x, ys)) & \to & \mathsf{ins}(x+1, \mathsf{inc}(ys)) & \\ & \mathsf{nats}(0, 0) & \to & \mathsf{ins}(0, \emptyset) & \\ & \mathsf{nats}(0, y+1) & \to & \mathsf{ins}(0, \mathsf{nats}(1, y+1)) & \\ & \mathsf{nats}(x+1, 0) & \to & \emptyset & \\ & \mathsf{nats}(x+1, y+1) & \to & \mathsf{inc}(\mathsf{nats}(x, y)) & \end{array}$$
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- Flexible framework for showing absence of infinite chains
- Originally developed for ordinary rewriting (Giesl *et. al.*, 2005)
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We show soundness of several DP processors for our class of rewrite systems:

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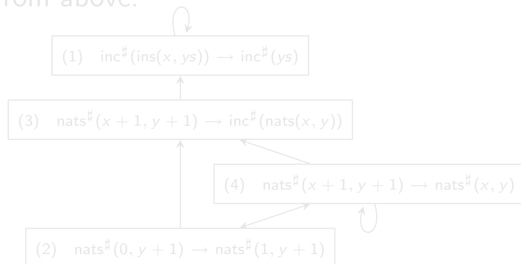

- Every infinite chain has an infinite tail that stays in some SCC
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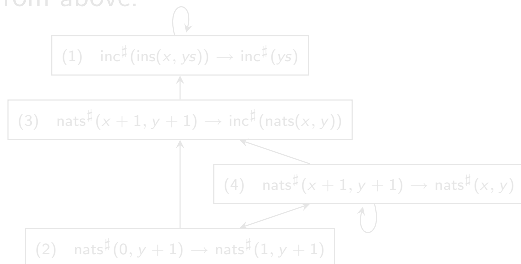

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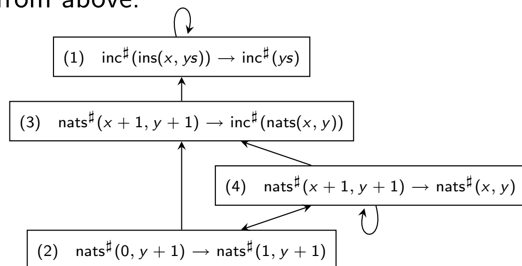

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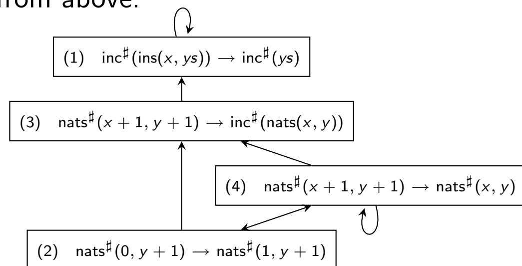

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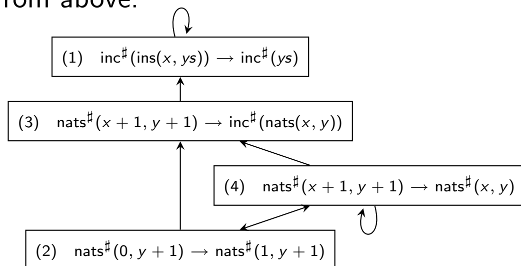

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Subterm Criterion

- Simple yet powerful DP processor (for ordinary rewriting due to Hirokawa & Middeldorp, 2004)
- We use the following strict subterm relations on (constrained) terms:
  - for s, t of sort $\mathtt{nat}$ we let $s[\![C]\!] \rhd_{\mathtt{nat}} t[\![C]\!]$ iff $C \Rightarrow s > t$ is $\mathcal{PA}$-valid
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Example

- For the DP problem

$$(\{\mathsf{inc}^\sharp(\mathsf{ins}(x, ys)) \to \mathsf{inc}^\sharp(ys)\}, \mathcal{R})$$

we use the simple projection $\pi(\mathsf{inc}^\sharp) = 1$. Then this DP problem is transformed into $(\emptyset, \mathcal{R})$ since

$$\pi(\mathsf{inc}^\sharp(\mathsf{ins}(x, ys))) = \mathsf{ins}(x, ys) \rhd_{\mathtt{univ}} ys = \pi(\mathsf{inc}^\sharp(ys))$$

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$$\begin{array}{lrcll}(\{ & \mathsf{nats}^\sharp(0, y+1) & \to & \mathsf{nats}^\sharp(1, y+1), & \\ & \mathsf{nats}^\sharp(x+1, y+1) & \to & \mathsf{nats}^\sharp(x, y) & \}, \mathcal{R})\end{array}$$

  we use the simple projection $\pi(\mathsf{nats}^\sharp) = 2$. Then this DP problem is transformed into $(\{\mathsf{nats}^\sharp(0, y+1) \to \mathsf{nats}^\sharp(1, y+1)\}, \mathcal{R})$ since

$$\pi(\mathsf{nats}^\sharp(0, y+1) = y + 1 \trianglerighteq_{\mathtt{nat}} 1 = \pi(\mathsf{nats}^\sharp(1, y+1))$$
$$\pi(\mathsf{nats}^\sharp(x+1, y+1)) = y + 1 \vartriangleright_{\mathtt{nat}} y = \pi(\mathsf{nats}^\sharp(x, y))$$

The resulting DP problem is handled by the dependency graph

$\mathcal{PA}$-polynomial Interpretations

- A $\mathcal{PA}$-polynomial interpretation assigns polynomials to function symbols such that
  - $\mathcal{P}ol(0) = 0$, $\mathcal{P}ol(1) = 1$, and $\mathcal{P}ol(x_1 + x_2) = x_1 + x_2$
  - $\mathcal{P}ol(f) \in \mathbb{N}[x_1, \ldots, x_n]$ if f has n arguments
  - $\mathcal{P}ol(f^\sharp) \in \mathbb{Z}[x_1, \ldots, x_n]$ if $f^\sharp$ has n arguments, where $\mathcal{P}ol(f^\sharp)$ is weakly increasing in all x_i that correspond to arguments that do not have sort `nat`
- Similar to the class of polynomial interpretations used in (Giesl *et. al.*, 2007) for innermost termination analysis of ordinary rewriting
- Gives rise to relations on constrained terms:
  - $s⟦C⟧ \succ_{\mathcal{P}ol} t⟦C⟧$ iff $C \Rightarrow [s]_{\mathcal{P}ol} \geq 0$ and $C \Rightarrow [s]_{\mathcal{P}ol} > [t]_{\mathcal{P}ol}$
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DP Processor Based on $\mathcal{PA}$-polynomial Interpretations

- Dependency pairs that are decreasing w.r.t. $\succ_{\mathcal{P}ol}$ cannot be used infinitely often in an infinite chain
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Example

- For the first imperative program we obtain the DP problem

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\begin{array}{rlll}
(\{ & \mathsf{eval}_0^\sharp(x,y) & \to & \mathsf{eval}_1^\sharp(x+1,1) & [\![x > 0]\!] \\
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\end{array}
$$

- Since terms of sort `nat` cannot be reduced it suffices to consider the dependency pairs
- First, we use the $\mathcal{PA}$-polynomial interpretation with $\mathcal{P}ol(\mathsf{eval}_0^\sharp) = x_1 + 1$ and $\mathcal{P}ol(\mathsf{eval}_1^\sharp) = x_1$ to remove the third dependency pair
- Using the dependency graph, the first dependency pair is removed
- Finally, we use the $\mathcal{PA}$-polynomial interpretation with $\mathcal{P}ol(\mathsf{eval}_1^\sharp) = x_1 - x_2$ to remove the second dependency pair since $x > y \Rightarrow x - y \geq 0$ and $x > y \Rightarrow x - y > x - (y+1)$ (Recall that we consider instantiations over $\mathbb{N}$ only)

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- We defined an expressive class of rewrite systems with semantic data structures and built-in natural numbers
- Both functional and imperative programs can be translated into this class of rewrite systems
- We extended the dependency pair framework for termination proving to this class of rewrite systems
- We presented powerful DP processors for this framework
- Future work:
  - Conditional rewriting (done)
  - Further DP processors
  - Sufficient conditions for confluence, sufficient completeness, etc.
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- Both functional and imperative programs can be translated into this class of rewrite systems
- We extended the dependency pair framework for termination proving to this class of rewrite systems
- We presented powerful DP processors for this framework
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    - Conditional rewriting (done)
    - Further DP processors
    - Sufficient conditions for confluence, sufficient completeness, etc.
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