Self-similar Sets, Dimensions and Kolmogorov Complexity

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Notation

Strings and Languages

Finite Alphabet $X = \{0, \ldots, r-1\}$, cardinality $|X| = r$

Finite strings (words): $w = x_1 \cdots x_l \in X^*$, $x_i \in X$, length $|w| = l$

Languages: $W \subseteq X^*$

Infinite strings (ω-words): $\xi = x_1 \cdots x_l \cdots \ \in X^\omega$

Prefixes of infinite strings: $\xi[0..n] \in X^*$, $\big|\xi[0..n]\big| = n$

ω-Languages: $F \subseteq X^\omega$

Language of prefixes : For $B \subseteq x^* \cup X^\omega$:
$\mathbf{pref}(B) := \{\eta[0..n] : n \in \mathbb{N} \wedge \eta \in B \wedge |\eta| \geq n\}$

1 Self-similar Sets

X^ω as Cantor space

Metric: $\rho(\eta,\xi) := \inf\{r^{-|w|} : w \sqsubset \eta \wedge w \sqsubset \xi\}$

Balls in (X^ω,ρ)**:** $w \cdot X^\omega = \{\eta : w \sqsubset \eta\}$

Diameter: $\operatorname{diam} w \cdot X^\omega = r^{-|w|}$

Open sets: $W \cdot X^\omega$

Closure: $\mathcal{C}(F) = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$

Property 1.1 $F \subseteq X^\omega$ *is closed if and only if* $\mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)$ *implies* $\xi \in F$.

Self-similar sets in Cantor space

Similarity: $\phi_w(\xi) := w \cdot \xi$

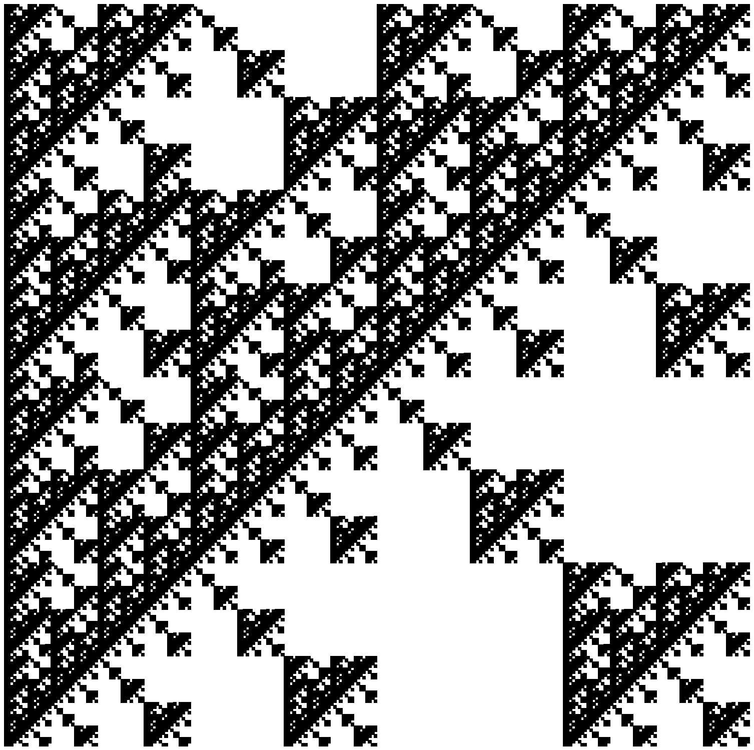
Defining Equation: $F = \bigcup_{w \in W} \phi_w(F) \qquad (*)$

Fixed point: maximal solution of **(*)**

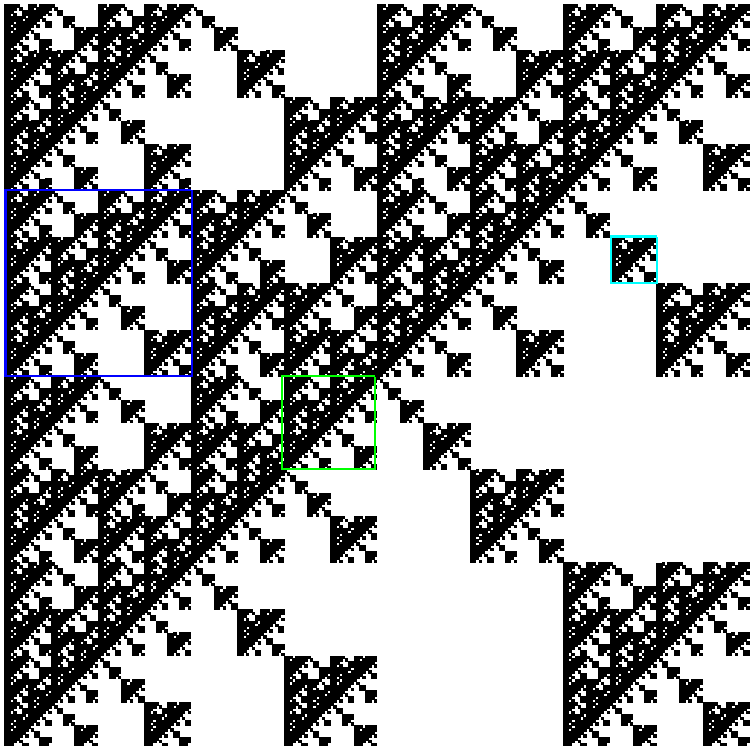
$W^\omega = \{w_1 \cdots w_i \cdots : w_i \in W \wedge |w_i| \geq 1 \text{ for } i \in \mathbb{N}\}$

Attractor: $\mathcal{C}(W^\omega)$

Property 1.2 *W^ω is closed in (X^ω, ρ) if W is finite.*



$$S_1 = SE \cdot S_3 \cup SW \cdot S_1 \cup NE \cdot S_1 \cup NW \cdot S_2$$
$$S_2 = SE \cdot S_2 \cup SW \cdot S_1 \cup NE \cdot S_3 \cup NW \cdot S_1$$
$$S_3 = SE \cdot S_1 \cup NW \cdot S_3$$



$$u = NE \cdot SE \cdot NW \cdot SE$$

$$w = NW \cdot SW$$

$$v = SW \cdot NE \cdot NE$$

$$\begin{aligned}
S_1 &= SE \cdot S_3 \cup SW \cdot S_1 \cup NE \cdot S_1 \cup NW \cdot S_2 \\
S_2 &= SE \cdot S_2 \cup SW \cdot S_1 \cup NE \cdot S_3 \cup NW \cdot S_1 \\
S_3 &= SE \cdot S_1 \cup NW \cdot S_3
\end{aligned}$$

2 Kolmogorov Complexity of Finite Words

Simple Algorithms Universal (optimal) algorithm: $\mathfrak{U}$

Complexity: $\mathsf{K}(w) := \mathsf{K}_{\mathfrak{U}}(w) = \inf\{|\pi| : \mathfrak{U}(\pi) = w\}$

Prefix Algorithms Partial-recursive functions $\mathfrak{C}' : X^* \to X^*$ where $\mathrm{dom}(\mathfrak{C}')$ is prefix-free.

Universal (optimal) algorithm: $\mathfrak{C}$

Complexity: $\mathsf{H}(w) := \mathsf{K}_{\mathfrak{C}}(w) = \inf\{|\pi| : \mathfrak{C}(\pi) = w\}$

Algorithm with length condition Partial-recursive functions $\mathfrak{A}' : X^* \times \mathbb{N} \to X^*$ where $\left|\mathfrak{A}'(\pi, n)\right| = n$.

Universal (optimal) algorithm: $\mathfrak{A}$

Complexity: $\mathsf{K}(w \mid |w|) := \inf\{|\pi| : \mathfrak{A}(\pi, |w|) = w\}$

Combinatorial Bounds

Definition 2.1 (Structure Function of W)

$$s_W(n) := \big|\{w : w \in W \wedge |w| = n\}\big| = \big|W \cap X^n\big|$$

Property 2.1 *If* $W \subseteq X^*$ *and* $s_W(n) \neq 0$ *then there is a* $w_n \in W \cap X^n$ *such that*

$$K(w_n \mid n) \geq \lceil \log_r s_W(|w_n|) \rceil \ .$$

Theorem 2.2 (Kolmogorov '70, *St.* '93) *If* $W \in \Sigma_1 \cup \Pi_1$ *then there is a* $c \in \mathbb{R}$ *such that, for all* $w \in W$,

$$K(w \mid |w|) \leq \log_r (1 + s_W(|w|)) + c \ .$$

If $W \in \Sigma_1 \cap \Pi_1$ *then there is a* $c \in \mathbb{R}$ *such that, for all* $w \in W$,

$$K(w) \leq \log_r \left(1 + \sum_{i=0}^{|w|} s_W(i)\right) + c \ .$$

Entropy of Languages [Shannon '49, Chomsky/Miller '58]

Definition 2.2

$$H_W := \limsup_{n\to\infty} \frac{\log_r(s_W(n)+1)}{n}$$

Corollary 2.3 $\alpha > H_W$ *implies* $\sum_{w\in W} r^{-\alpha\cdot|w|} < \infty$ *and* $\sum_{w\in W} r^{-\alpha\cdot|w|} < \infty$ *implies* $\alpha \geq H_W$.

Corollary 2.4 *If* $W \in \Sigma_1 \cup \Pi_1$ *then* $K(w) \leq \mathbf{H}_W \cdot |w| + o(|w|)$*, for* $w \in W$.

Definition 2.3 $W_\alpha := \{w : K(w) < \alpha \cdot |w|\}$

Lemma 2.5 $H_{W_\alpha} \leq \alpha$ *for* $0 \leq \alpha \leq 1$

If $\alpha \in [0,1]$ *is left-computable then* $W_\alpha \in \Sigma_1$.

3 Dimensions of ω-languages

Minkowski or box-counting dimensions

$$\underline{\dim}_{\mathbf{B}} F := \liminf_{n\to\infty} \frac{\log_r(s_{\mathbf{pref}(F)}(n)+1)}{n}$$

$$\overline{\dim}_{\mathbf{B}} F := \limsup_{n\to\infty} \frac{\log_r(s_{\mathbf{pref}(F)}(n)+1)}{n} = \mathbf{H}_{\mathbf{pref}(F)}$$

Remark: $s_{\mathbf{pref}(F)}(n)$ is the number of balls of diameter r^{-n} which are necessary to cover F.

Property 3.1

$$\underline{\dim}_{\mathbf{B}} F = \underline{\dim}_{\mathbf{B}} \mathcal{C}(F)$$

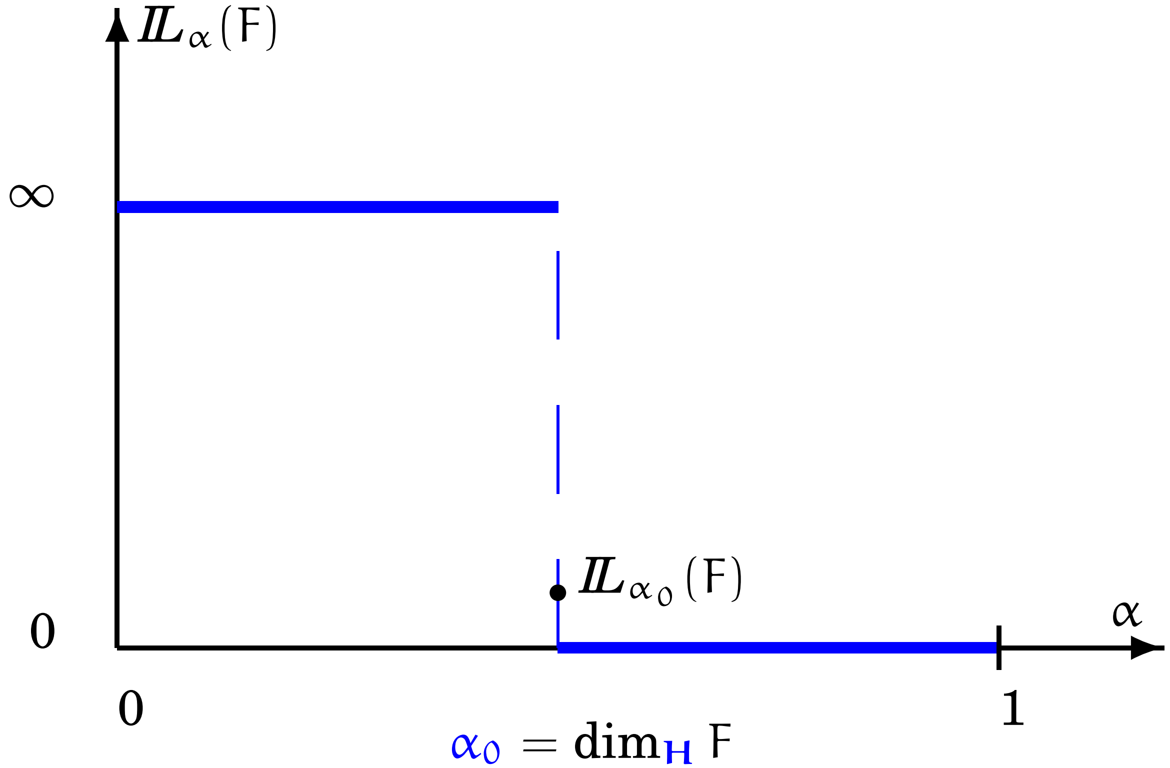
$$\overline{\dim}_{\mathbf{B}} F = \overline{\dim}_{\mathbf{B}} \mathcal{C}(F)$$

$$E \subseteq F \to \underline{\dim}_{\mathbf{B}} E \leq \underline{\dim}_{\mathbf{B}} F$$

$$\overline{\dim}_{\mathbf{B}} E \cup F = \max\{\overline{\dim}_{\mathbf{B}} E, \overline{\dim}_{\mathbf{B}} F\}$$

Hausdorff measure

$$\mathbb{L}_\alpha(F) := \lim_{l\to\infty} \inf\Big\{ \sum_{w\in W} r^{-\alpha\cdot|w|} : F \subseteq W\cdot X^\omega \wedge \forall w(w\in W \to |w| \geq l)\Big\}$$



Hausdorff dimension

$$\dim_{\mathbf{H}} F = \sup\{\alpha : I\!L_\alpha(F) = \infty\} = \inf\{\alpha : I\!L_\alpha(F) = 0\}$$

Packing or modified upper box-counting dimension

$$\dim_{\mathbf{P}} F := \overline{\dim}_{\mathbf{MB}} F = \inf\{\sup_{i\in\mathbb{N}} \overline{\dim}_{\mathbf{B}} F_i : \bigcup_{i\in\mathbb{N}} F_i \supseteq F\}$$

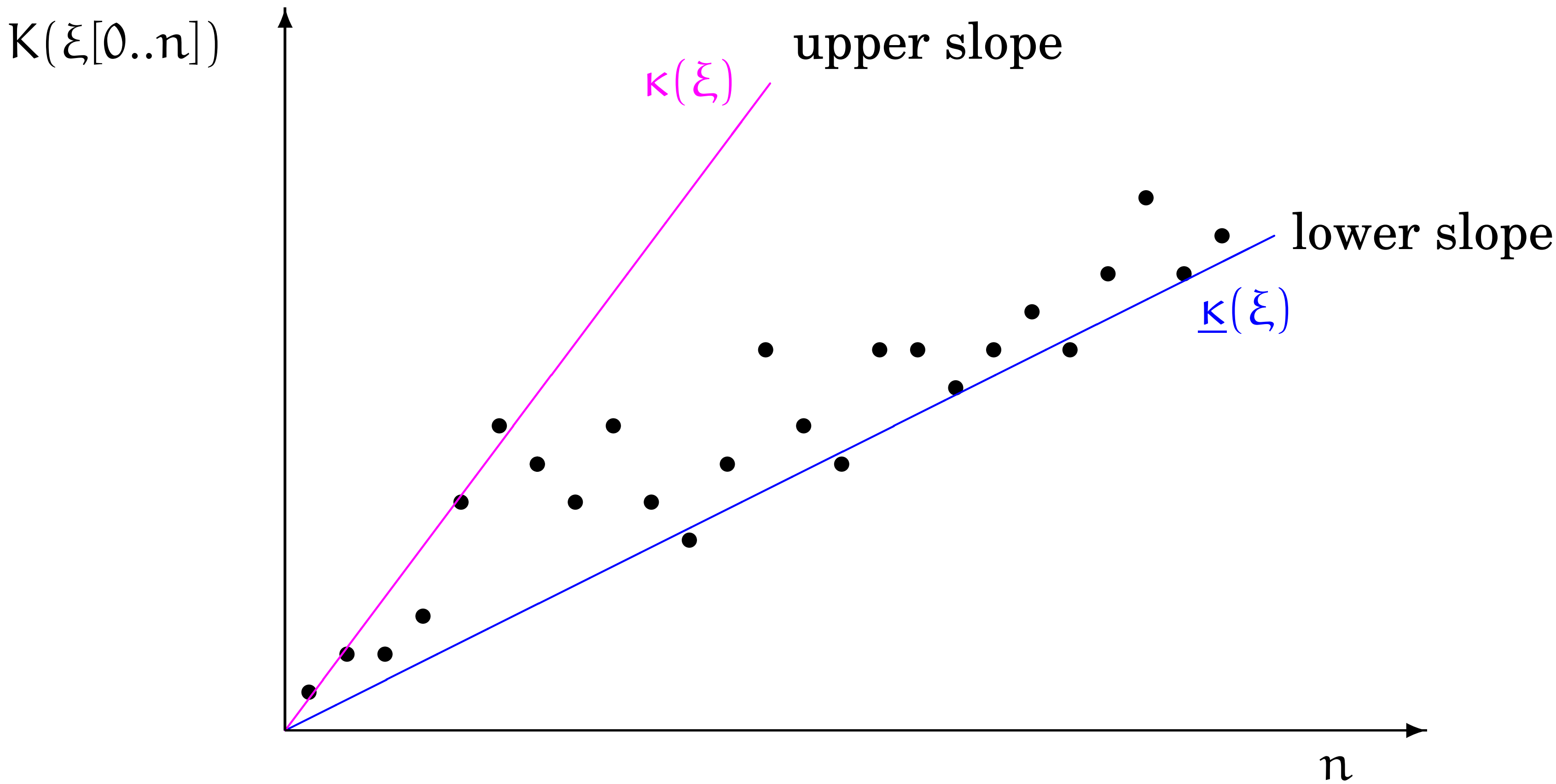
Proposition 3.2 *1. If* $\dim \in \{\dim_{\mathbf{H}}, \dim_{\mathbf{P}}\}$ *then*

$$\dim \bigcup_{i\in\mathbb{N}} F_i = \sup\{\dim F_i : i \in \mathbb{N}\}$$

2. $\dim_{\mathbf{H}} F = \inf\{\mathbf{H}_W : \forall\xi(\xi \in F \rightarrow |\mathbf{pref}(\xi) \cap W| = \infty\}$

3. $\dim_{\mathbf{P}} F = \inf\{\mathbf{H}_W : \forall\xi(\xi \in F \rightarrow |\mathbf{pref}(\xi) \setminus W| < \infty\}$

4 Kolmogorov Complexity of Infinite Words



First order approximation
upper and lower Kolmogorov complexity

$$\kappa(\xi) := \limsup_{n\to\infty} \frac{K(\xi[0..n])}{n} \qquad \kappa(F) := \sup\{\kappa(\xi) : \xi \in F\}$$

$$\underline{\kappa}(\xi) := \liminf_{n\to\infty} \frac{K(\xi[0..n])}{n} \qquad \kappa(F) := \sup\{\underline{\kappa}(\xi) : \xi \in F\}$$

Random ω-words have $\underline{\kappa}(\xi) = 1$.

Recursive (computable) ω-words have $\kappa(\xi) = 0$.

Expansions of Liouville numbers have $\underline{\kappa}(\xi) = 0$.

Other ω-words:

e.g. $\xi' := x_1 0 x_2 0 \ldots$ has $\kappa(\xi') = \frac{1}{2}\kappa(\xi)$ when $\xi := x_1 x_2 \ldots$.

Kolmogorov Complexity of Infinite Words

Brudno	'74, '78	Connection to topological entropy
Daley	'74, '75	Complexity of individual ω-words
Ryabko	'84, '86	Connection to Hausdorff dimension
St.	'81, '89, '93, '98	Connection to box-counting dimension and Hausdorff measure
Lutz et al., Hitchcock	since 2000	Introduction of constructive dimension (equals Kolmogorov complexity) using betting strategies (martingales)

Kolmogorov complexity and dimensions
Lower bounds

Theorem 4.1 (Lower Bound, Ryabko '86, Athreya et al. '04)

$$\dim_{\mathbf{H}} F \leq \sup\{\underline{\kappa}(\xi) : \xi \in F\} \textit{ and}$$

$$\dim_{\mathbf{P}} F \leq \sup\{\kappa(\xi) : \xi \in F\}$$

Theorem 4.2 (Refined Lower Bound, *St. '93*, Calude et al. '05)
Let $F \subseteq X^{\omega}$, $\mathbb{L}_{\alpha}(F) > 0$ *and* $\sum_{n \in \mathbb{N}} r^{-f(n)} < \infty$ *for* $f : \mathbb{N} \to \mathbb{N}$.

1. *Then there is a* $\xi \in F$ *such that* $K(\xi[0..n] \mid n) \geq_{\mathrm{ae}} \alpha \cdot n - f(n)$.
2. *Then there is a* $\xi \in F$ *such that* $H(\xi[0..n]) \geq_{\mathrm{ae}} \alpha \cdot n - O(1)$.

Kolmogorov complexity
Characterisation via entropy

$$\{\zeta : \underline{\kappa}(\zeta) \le \gamma\} \subseteq \bigcap_{\gamma<\alpha} \{\xi : |\mathbf{pref}(\xi) \cap W_\alpha| = \infty\} \text{ and}$$

$$\{\zeta : \kappa(\zeta) \le \gamma\} \subseteq \bigcap_{\gamma<\alpha} \{\xi : |\mathbf{pref}(\xi) \setminus W_\alpha| < \infty\}$$

Lemma 4.3 (Kolmogorov '70, *St. '93*, Hitchcock '05)

$$\begin{aligned}
\underline{\kappa}(F) &= \inf\{H_W : W \in \Sigma_1 \cup \Pi_1 \wedge \forall\xi(\xi \in F \to |\mathbf{pref}(\xi) \cap W| = \infty)\} \\
&= \inf\{H_W : W \in \Sigma_1 \wedge \forall\xi(\xi \in F \to |\mathbf{pref}(\xi) \cap W| = \infty)\} \\
\kappa(F) &= \inf\{H_W : W \in \Sigma_1 \cup \Pi_1 \wedge \forall\xi(\xi \in F \to |\mathbf{pref}(\xi) \setminus W| < \infty)\} \\
&= \inf\{H_W : W \in \Sigma_1 \wedge \forall\xi(\xi \in F \to |\mathbf{pref}(\xi) \setminus W| < \infty)\}
\end{aligned}$$

5 Complexity Bounds for self-similar sets (ω-power languages)

ω-power languages: $W^\omega := \{w_1 \cdots w_i \cdots : w_i \in W \wedge |w_i| > 0\}$

Proposition 5.1

$$\begin{aligned} \dim_{\mathbf{H}} W^\omega &= \mathsf{H}_{W^*} \quad \textit{and} \\ \dim_{\mathbf{P}} \mathcal{C}(W^\omega) &= \overline{\dim}_{\mathbf{B}} W^\omega = \mathsf{H}_{\mathbf{pref}(W^*)} \end{aligned}$$

Recall:

$$\dim_{\mathbf{H}} F \le \dim_{\mathbf{P}} F \le \overline{\dim}_{\mathbf{B}} F$$

Lemma 5.2 *For* $W \subseteq X^*$ *let* $(m_i)_{i\in\mathbb{N}}$ *be a family of natural numbers and let* $(v_i)_{i\in\mathbb{N}}$ *be a family of words in* $\mathbf{pref}(W^*)$ *such that* $|v_i| < |v_{i+1}|$, $m_i/|v_i| < m_{i+1}/|v_{i+1}|$ *and* $K(v_i \mid |v_i|) \geq m_i$. *Then there is a* $\xi \in W^\omega$ *such that* $\kappa(\xi) \geq \sup\{m_i/|v_i| : i \in \mathbb{N}\}$.

Theorem 5.3 (Lower κ-bound)

$$\kappa(\mathcal{C}(W^\omega)) = \kappa(W^\omega) = \max\{\kappa(\xi) : \xi \in W^\omega\} \geq \overline{\dim}_{\mathbf{B}}\, W^\omega$$

Corollary 5.4 (Exact κ-bound)

$$\sup\{\kappa(\xi) : \xi \in \mathcal{C}(W^\omega)\} = \overline{\dim}_{\mathbf{B}}\, W^\omega \text{ if } W \in \Sigma_1 \ .$$

Theorem 5.5 (Exact $\underline{\kappa}$-bound) *If* $W \in \Sigma_1 \cup \Pi_1$ *then*

$$\dim_{\mathbf{H}} W^\omega = \sup\{\underline{\kappa}(\xi) : \xi \in W^\omega\} \ .$$

6 Bounds for Regular ω-languages

Regular ω-languages: $F = \bigcup_{i=1}^{n} V_i \cdot W_i^{\omega}$ where all languages V_i, W_i are regular.

Proposition 6.1 *1. If $W \subseteq X^*$ is regular then*
$$\dim_{\mathbf{H}} W^{\omega} = \overline{\dim}_{\mathbf{B}} W^{\omega}$$

2. If $F \subseteq X^{\omega}$ is regular and closed then $\dim_{\mathbf{H}} F = \overline{\dim}_{\mathbf{B}} F$.

Corollary 6.2 *If $F \subseteq X^{\omega}$ is regular then* $\dim_{\mathbf{H}} F = \dim_{\mathbf{P}} F$.

Theorem 6.3 (Linear upper Bound, *St. '81*) *Let $F \subseteq X^{\omega}$ be regular and* $\dim_{\mathbf{H}} F > 0$. *Then*
$$\forall \xi (\xi \in F \rightarrow \exists c_{\xi} (c_{\xi} \in \mathbb{R} \wedge K(\xi[0..n]) \leq_{ae} \dim_{\mathbf{H}} F \cdot n + c_{\xi})).$$

Lower bound and oscillation

Theorem 6.4 (Lower Bound) *Let* $F \subseteq X^\omega$ *be nonempty and regular and let* $f : \mathbb{N} \to \mathbb{N}$ *satisfy* $\sum_{n\in\mathbb{N}} r^{-f(n)} < \infty$. *Then there are* $\xi, \zeta \in F$ *such that*

$$\begin{aligned} K(\xi[0..n] \mid n) &\geq_{ae} \dim_H F \cdot n - f(n)\,, \text{ and} \\ H(\zeta[0..n]) &\geq_{ae} \dim_H F \cdot n - O(1)\,. \end{aligned}$$

Theorem 6.5 (Oscillation) *Let* $F \subseteq X^\omega$ *be nonempty and regular. If* $f : \mathbb{N} \to \mathbb{N}$ *is recursive and satisfies* $\sum_{n\in\mathbb{N}} r^{-f(i)} = \infty$ *then*

$$\forall \xi \big(\xi \in F \to K(\xi[0..n] \mid n) \leq_{io} \dim_H F \cdot n - f(n) \big)\,.$$