Points on Computable Curves

The computable analyst’s traveling salesman theorem

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Computable curves

- A computable curve is a function $\Upsilon: [0,1] \rightarrow R^2$ such that Υ is computable

2 is a simplification for this talk

(bitwise definition: from an approximation of t we compute an approximation of $\Upsilon(t)$)

- A computable parametrization is enough
- No injectivity condition

Rectifiable computable curves

- There are “space filling” computable curves (of course infinite length)
- We are interested in curves of finite length

  where can you get in finite time through a computable path?

The questions

- Which sets in R^2 can be traversed by a computable curve of finite length?

  $A \subseteq \Upsilon([0,1])$ for some Υ computable and rectifiable?

- Which points belong to computable curves of finite length?

  Which $x \in \Upsilon([0,1])$ for some Υ computable and rectifiable?

This set is PCC

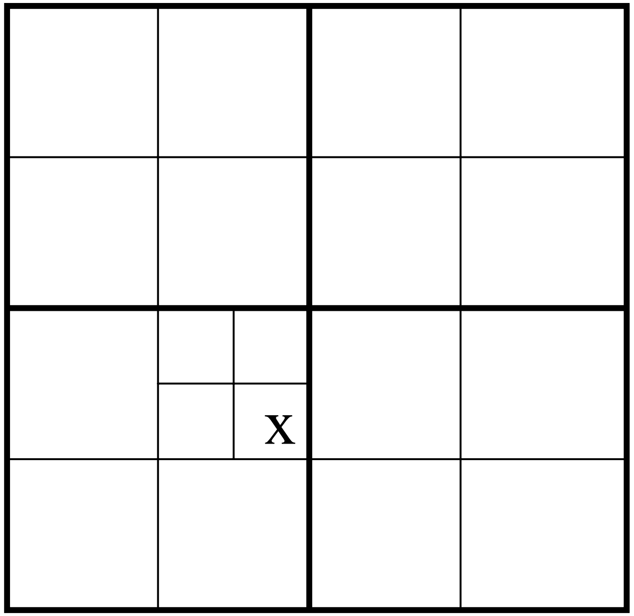
Something to do with randomness?

- Consider points (x,y) in R^2
- Random points are those for which their corresponding binary sequence is random

Sequence corresponding to a point

- (x,y) in R^2

00	01
10	11



100111…

+ integer part

Constructive dimension

- Constructive dimension is a generalization of Hausdorff dimension
- The constructive dimension of a point can be characterized as

$$\dim(x,y) = \liminf_n \frac{K(x,y[1..2n])}{n}$$

- Random points have dimension 2

Where are nonrandom points?

- All points in PCC have constructive dimension at most 1,

  $\liminf_n K(x,y[1..2n])/n \leq 1$

- Are all points with dimension at most 1 in PCC?

  → NO

About PCC

- There are points out of PCC with dimension any arbitrary value in [0,1]
- PCC is arc-connected (by computable curves of finite length)
- PCC is dense (it contains all computable points)
- PCC has Hausdorff dimension 1 (and constructive dimension 1)

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Analyst’s traveling salesman theorem

- Jones (1990) and Okikiolu (1991) prove a characterization of a set K contained in a curve of finite length

- The characterization is connected to the way the set can be approximated

Jones beta-number

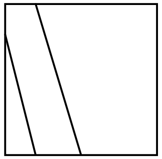
- Let K be a bounded set in R^2
- We cut the space in dyadic squares

  $Q= [a2^{-m}, (a+1)2^{-m}) \times [b2^{-m}, (b+1)2^{-m})$

  $a, b \in Z$

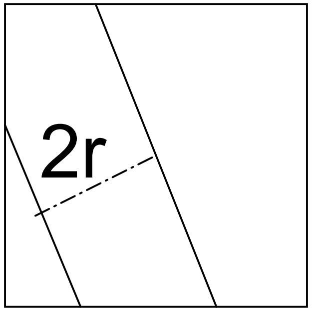
  $l(Q)=2^{-m}$

- We look at the cylinders covering $K \cap 3Q$



Jones beta-number

- Given a set K and a dyadic square Q r(Q) is the smallest radius of a cylinder covering $K \cap 3Q$



$$\beta_Q(K) = r(Q) / l(Q)$$

$$\beta^2(K) = \sum_Q \beta_Q(K)^2 \, l(Q)$$

Jones theorem

- Let $K \subseteq R^2$ be a bounded set. Then K is contained in some rectifiable curve if and only if $\beta^2(K) < \infty$

Proof of Jones theorem

- If part: complicated construction of a curve, chooses freely points in K with certain properties

- Only if, Okikiolu: clever geometric argument

A computable version??

- We first have to replace Jones beta-number by a data structure that can be required to be computable
- We assign a cylinder to each dyadic square

Constraints

- A **constraint** is a function
  Υ: dyadic squares → cylinders

- The set permitted by Υ is:

$$\kappa(\Upsilon) =$$

$$\cap_Q[(\Upsilon(Q) \cap (3Q)^0) \;\cup\; (R^2 - (3Q)^0)]$$

Computable beta-number

- Given a constraint Υ: dyadic squares → cylinders, the corresponding radius are given by

  ρ: dyadic squares → [0,1]

- $\beta_Q(\Upsilon) = \rho(Q) / l(Q)$

- $\beta^2(\Upsilon) = \sum_Q \beta_Q(\Upsilon)^2\, l(Q)$

Computable analyst's TS theorem

- Let $K \subseteq R^2$ be a bounded set.

Then K is contained in some **computable** rectifiable curve

if and only

there is a **computable** constraint β such that $K \subseteq \kappa(\Upsilon)$ and $\beta^2(\Upsilon) < \infty$

Proof

- The if part is an elaborated algorithm to construct a curve from the constraint using a farthest insertion principle and amortized analysis for the case when this is not possible

- Notice that we have no access to the points in K, only an algorithm that computes the constriction

  → no access to $\kappa(\Upsilon)$

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- **Submission Deadline: 9 February 2006**
- LNCS proceedings

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